Free field theory at null infinity and white noise calculus: a BMS invariant dynamical system

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Abstract. In the context of asymptotically flat spacetimes we exploit techniques proper either of white noise analysis either of dynamical systems in order to develop the Lagrangian and the Hamiltonian approach to a BMS invariant field theory at null infinity.

1 Introduction

The quest to understand, to clarify and, to a certain extent, also to develop more in detail field theory over curved backgrounds both at a classical and at a quantum level strongly relied in the last decade on the holographic principle.

Originally introduced by 't Hooft in [1] to study black hole backgrounds and the related information paradox, it still lacks nowadays of a mathematically rigorous and universally accepted definition in a general framework. Nonetheless the most promising though rather heuristic and, at the same time, demanding statement is the following: any field theory living on a D-dimensional manifold M - possibly including gravity - can be described by means of a suitable second field theory living on a codimension one submanifold of M.

From an abstract point of view such assertion is rather counterintuitive and at the same time revolutionary since it states that the degrees of freedom encoding the information of a physical system evolving on a given background can be stored on a properly chosen lower dimensional region. On a practical ground, instead, even though one is inclined to believe in such a conjecture, it is straightforward to realize that the above formulation does not provide any concrete mean or hint on how to effectively implement the holographic paradigm.

Amending such lack has been one of the main guideline in theoretical high-energy physics and in mathematical physics research for the past few years; at present a cornerstone is represented by the widely accepted realization of holography for field theories living in asymptotically anti-de-Sitter spacetimes *i.e.* solutions to Einstein's field equation with a negative cosmological constant. Such a breakthrough (see [2] for an old but still complete review), originates from the so-called AdS/CFT correspondence; it hypothesize the existence of a 1:1 correspondence between a type IIB superstring theory living in the bulk of $\mathrm{AdS}_5 \times S^5$ and a $SU(N)$ super Yang-Mills field theory living on its conformal boundary. Nonetheless, in order to catch some glimpses of its complexity, one should notice that, at a mathematical level, a rigorous and complete proof of

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the Maldacena conjecture is still only partially available thanks to an analysis of free field theory in asymptotically AdS spacetimes within the framework of the algebraic formulation of quantum field theory [3, 4].

In this paper we will not directly address the above briefly discussed topos whereas we will focus on a related though different problem namely if it is also possible to realize the holographic principle for field theories living on backgrounds solutions to Einstein's field equations with vanishing cosmological constant. To be more precise, we will deal with those spacetimes which are asymptotically flat at future (or past) null infinity *i.e.*, in a sense better specified in the next section, they admit a conformal completion; thus they can be endowed with a natural notion of boundary representing, as in the AdS/CFT correspondence, the codimension one submanifold where to encode the data from a bulk field theory.

Since the prototype of such class of manifolds is Minkowski spacetime, from a physical perspective, there is a strong interest in such a line of research since one could hope to concretely exploit holography to enhance our comprehension of quantum field theory over curved background and to eventually shed some light on some unsolved puzzles of quantum field theory over a flat background. From a mere mathematical perspective we will instead show how the development of a classical field theory at null infinity, as an holographic image of a suitable bulk counterpart, shows a deep-rooted and a priori unpredictable connection with recently developed techniques of functional analysis such as white noise calculus. Often binded to play a somehow ancillary role in classical or quantum field theory (see [5] for a recent application or a [6] and references therein for a more detailed analysis), this last mentioned framework represents the natural and the main machinery underlying the concrete development of a mathematically rigorous field theory at null infinity thus providing a further interesting motivation for this kind of research (see also [7] for a preliminary analysis leading in this direction).

Nonetheless "holography in asymptotically flat spacetimes", as a whole, is not in its childhood since a series of different papers appeared in the last few years discussing the problem from different points of view (see [8] and the recent [9] where the concept of holography is geometrically intertwined with spatial infinity instead of null infinity). In particular this paper can be placed along the lines of [10, 11] where it has been first explored and stated that a concrete realization of 't Hooft proposal not only should be developed at null infinity but it also must necessarily deal with a throughout analysis of the Bondi-Metzner-Sachs (BMS) group. As we will carefully discuss in the next sections, this is an infinite dimensional preferred subgroup of the diffeomorphism group of future (or past) null infinity which can be endowed with the structure of a semidirect product between the proper orthocronous component of the Lorentz group and the set of smooth functions over the 2-sphere seen as an abelian group under addition.

To summarize, the rationale we advocate is the following: it possible to holographically encode the data of a field theory living on an asymptotically flat spacetime in a BMS invariant free field theory at null infinity. At present,

within this respect, several progresses have been achieved; as a starting point we have exploited group theoretical techniques in [10] to classify and to construct, by means of Mackey theory of induction, the irreducible (and unitary) representations of the BMS group and the related induced or canonical wave functions. From a physical perspective, these maps - discussed in section 3 - do represent the set of all possible dynamically allowed configurations of the free fields of the theory and, together with their covariant counterpart, they also allow to fully characterize the dynamically allowed configurations for a BMS invariant field theory at null infinity (see [7, 13]). Furthermore these results have been recast in terms of an holographic correspondence in [13] where, exploiting the algebraic formulation of quantum field theory and the related operator algebra techniques, several "holographic theorems" have been proved. Between them, one of the key achievement consisted in showing the existence of a 1:1 correspondence between a massless scalar field conformally coupled to gravity in any asymptotically flat, globally hyperbolic spacetime and a BMS invariant induced wave function intrinsically defined at null infinity[2]

Nonetheless the whole approach advocated in [13] suffers of two main drawbacks; the first, which will not be discussed in this paper, concerns bulk massive fields: even in globally hyperbolic spacetimes, there is no known way to coherently project a solution of the Klein-Gordon equation with $m \neq 0$ to a smooth function on future (or past) null infinity. Consequently a "geometrical machinery" projecting such bulk data on the boundary is far from being completely constructed. The second deficiency, already sketched in the previous papers, consisted in a complete lack of interactions in the boundary theories and, in particular, (Yang-Mills) gauge theories were not at all taken into account. Such a deficency can be traced back to the overall approach that the infinite dimensional nature of the BMS group forced us to take *i.e.* Wigner programme. This analysis provides a construction of the relevant free fields and of their equations of motion without deriving the latter from a variational principle. Thus, at present, there is no real notion whether the BMS field theory admits a Lagrangian formulation and more importantly an Hamiltonian formulation over a suitably chosen symplectic space. From a point of view of interactions and in particular of gauge interactions, which are our ultimate goal in an holographic analysis, it is known that it possible to rigorously construct the coupling between a free field theory and gauge fields by means of a symplectic deformation of the free Hamiltonian system (see [14, 15] and in particular [16]). Wishing ultimately to follow a similar road in the framework of BMS field theory, our aim in this manuscript is to cover the first part of the above sketched programme, namely to identify if an Hamiltonian system can be associated with a BMS free field, leaving to a future paper the symplectic deformation leading to gauge coupling. On an operative ground we will focus our attention on the "working example"

[2]A further interesting result, though not directly connected with the aims of this paper, consists on the identification in [13] of a preferred algebraic BMS invariant vacuum state at null infinity which can be suitably pulled-back in the bulk coinciding, in a flat background, with the usual Minkowski vacuum state. Furthermore it also been recently proved the uniqueness of such a state (see [12] for the demonstration and for a discussion of the main properties.)

of the massive and massless real scalar field and we will follow a two-step road. In the first part we will study in detail the concept of a BMS covariant wave function which, although it appeared in the previously cited papers, has always played a sort of puppet role thus never being really carefully studied. To this avail, as we will outline mainly in section 4, we will require specific functional techniques such as white noise noise analysis which, thus, will be also covered with some care for sake of completeness. As a second step, in this paper, we shall consider the BMS counterpart of the real scalar field equations of motion and we will solve the related inverse Lagrangian problem eventually also proving that an Hamiltonian description can as well be coherently formulated.

As a side remark we wish also to point out that our attempt to construct a dynamical system at null infinity has been preceded by a now more than twenty five years old attempt by Ashtekar and Streubel in [17]. In this paper the authors introduced a suitable Fréchet space as the configuration space for a smooth scalar field living at null infinity and they exploit a theorem by Chernoff and Marsden from [18] in order to construct a BMS invariant symplectic phase space and an associated Hamiltonian phase space. Nonetheless the whole approach does not deal with the intrinsic "Wigner-like" formulation of a BMS invariant free field theory on null infinity and we shall comment further on this topic in the conclusions

Outline of the paper: The paper is divided, beside the introduction and the conclusions, in four main sections.

In the next one we will briefly sketch the geometrical concept underlying an asymptotically flat spacetime at future (or past) null infinity and we will introduce the key group-theoretical notion of the BMS group. Besides the recasting of already known concepts, the main issue of the whole section will be the introduction of an alternative and novel demonstration of a result partially exploited in [13, 19] *i.e.* we will prove that the BMS group and, more importantly, its abelian ideal are nuclear Lie groups.

Section 3 will briefly rephrase within the framework of fiber bundles, both the irreducible representations for the BMS group and the set of induced wave functions. In particular we will discuss more in detail the key "working" examples of the massless and the massive scalar field and we will develop an alternative way to introduce the Casimir invariant which plays the role of the mass for a field at null infinity.

In section 4, the main one of the paper, we will introduce the so-called covariant fields and we will show why and how the infinite dimensional nature of the BMS group forces us to introduce and fully exploit the powerful techniques of white noise calculus. Within this framework we will develop the key functional space where BMS field theory is defined in particular discussing the "BMS counterpart" of the Schwartz space of rapidly decreasing test functions and of distributions for a BMS invariant field theory. At the end of the section we will also provide a demonstration of the Wigner programme in this specific framework and the equations of motion for the BMS fields will be introduced as suitable operators on the above mentioned functional space.

Eventually in the fifth and last section we will start from this latter result and we will solve the inverse Lagrangian problem. Exploiting an old analysis due to Gotay and Nester on presymplectic Lagrangian systems we will also prove that the Lagrangian itself is almost regular and thus it is still possible to construct an associated Hamiltonian system.

2 Asymptotically flat spacetimes and the BMS group

Throughout this paper we will refer to a *spacetime* as a four-dimensional smooth (Hausdorff second countable) manifold M equipped with a Lorentzian metric g assumed to be everywhere smooth; finally M is supposed to be time orientable and time oriented. A *vacuum spacetime* is a spacetime satisfying vacuum Einstein equations.

We adopt the notion of *asymptotically flat at future null infinity* vacuum spacetime presented in [20] i.e. a smooth spacetime (M, g) is called *asymptotically flat vacuum spacetime at null infinity* if there is a second smooth spacetime $(\tilde{M}, \tilde{g})$ such that M turns out to be an open submanifold of $\tilde{M}$ with boundary $\Im \subset \tilde{M}$. $\Im$ is an embedded submanifold of $\tilde{M}$ satisfying $\Im \cap \tilde{J^-}(M) = \emptyset$. $(\tilde{M}, \tilde{g})$ is required to be strongly causal in a neighborhood of $\Im$ and it must hold $\tilde{g}\restriction_M = \Omega^2 \restriction_M g\restriction_M$ where $\Omega \in C^\infty(\tilde{M})$ is strictly positive on M. On $\Im$ one must have $\Omega = 0$ and $d\Omega \neq 0$. Moreover, defining $n^a := \tilde{g}^{ab}\partial_b\Omega$, there must be a smooth function, ω, defined in $\tilde{M}$ with $\omega > 0$ on $M \cup \Im$, such that $\tilde{\nabla}_a(\omega^4 n^a) = 0$ on $\Im$ and the integral lines of $\omega^{-1}n$ are complete on $\Im$. Finally the topology of each set $\Im^\pm$ must be that of $\mathbb{S}^2 \times \mathbb{R}$. $\Im$ is called *future null infinity* of M.
It is possible to make stronger the definition of asymptotically flat spacetime by requiring asymptotic flatness at both null infinity – including the *past* null infinity $\Im^-$ defined analogously to $\Im$ – and *spatial* infinity, given by a special point in $\tilde{M}$ indicated by i^0. The complete definition is due to Ashtekar (see Chapter 11 in [20] for a general discussion). We stress that the results presented in this work do not require such a stronger definition: for the spacetimes we consider existence of $\Im$ is fully enough.

Considering an asymptotically flat spacetime, the metric structures of $\Im^+$ are affected by a *gauge freedom* due the possibility of changing the metric $\tilde{g}$ in a neighborhood of $\Im^+$ with a factor ω smooth and strictly positive. It corresponds to the freedom involved in transformations $\Omega \to \omega\Omega$ in a neighborhood of $\Im^+$. The topology of $\Im^+$ (which is that of $\mathbb{R} \times \mathbb{S}^2$) as well as the differentiable structure are not affected by the gauge freedom. Let us stress some features of this extent. Fixing Ω, $\Im^+$ turns out to be the union of future-oriented integral lines of the field $n^a := \tilde{g}^{ab}\tilde{\nabla}_b\Omega$. This property is, in fact, invariant under gauge transformation, but the field n depends on the gauge. For a fixed asymptotically flat vacuum spacetime (M, g), the manifold $\Im^+$ together with its degenerate metric $\tilde{h}$ induced by $\tilde{g}$ and the field n on $\Im^+$ form a triple which, under gauge

transformations $\Omega \to \omega\Omega$, transforms as

$$\Im^+ \to \Im^+ \,, \quad \tilde{h} \to \omega^2 \tilde{h} \,, \quad n \to \omega^{-1} n \,. \tag{1}$$

If C denotes the class containing all of the triples $(\Im^+, \tilde{h}, n)$ transforming as in (1) for a fixed asymptotically flat vacuum spacetime (M, g), there is no general physical principle which allows one to select a preferred element in C. Conversely, C is *universal* for all asymptotically flat vacuum spacetimes in the following sense. If C_1 and C_2 are the classes of triples associated respectively to (M_1, g_1) and (M_2, g_2) there is a diffeomorphism $\gamma : \Im^+_1 \to \Im^+_2$ such that for suitable $(\Im^+_1, \tilde{h}_1, n_1) \in C_1$ and $(\Im^+_2, \tilde{h}_2, n_2) \in C_2$,

$$\gamma(\Im^+_1) = \Im^+_2 \,, \quad \gamma^* \tilde{h}_1 = \tilde{h}_2 \,, \quad \gamma^* n_1 = n_2 \,.$$

The proof of this statement relies on the following nontrivial result [20]. For whatever asymptotically flat vacuum spacetime (M, g) (either (M_1, g_1) and (M_2, g_2) in particular) and whatever initial choice for Ω_0, varying the latter with a judicious choice of the gauge ω, one can always fix $\Omega := \omega\Omega_0$ in order that the metric $\tilde{g}$ associated with Ω satisfies

$$\tilde{g}\restriction_{\Im^+} = -2du\, d\Omega + d\Sigma_{\mathbb{S}^2}(x_1, x_2) \,. \tag{2}$$

This formula uses the fact that in a neighborhood of $\Im^+$, (u, Ω, x_1, x_2) define a meaningful coordinate system. $d\Sigma_{\mathbb{S}^2}(x_1, x_2)$ is the standard metric on a unit 2-sphere (referred to arbitrarily fixed coordinates x_1, x_2) and $u \in \mathbb{R}$ is nothing but an affine parameter along the *complete* null geodesics forming $\Im^+$ itself with $n = \partial/\partial u$. In these coordinates $\Im^+$ is just the set of the points with $u \in \mathbb{R}$, $(x_1, x_2) \in \mathbb{S}^2$ and, no-matter the initial spacetime (M, g) (either (M_1, g_1) and (M_2, g_2) in particular), one has finally the triple $(\Im^+, \tilde{h}_B, n_B) := (\mathbb{R} \times \mathbb{S}^2, d\Sigma_{\mathbb{S}^2}, \partial/\partial u)$.

Definition 2.1. The **Bondi-Metzner-Sachs (BMS) group**, G_{BMS} [17, 21, 22, 23], is the group of diffeomorphisms of $\gamma : \Im^+ \to \Im^+$ which preserves the universal structure of $\Im^+$, i.e. $(\gamma(\Im^+), \gamma^*\tilde{h}, \gamma^* n)$ differs from $(\Im^+, \tilde{h}, n)$ at most by a gauge transformation (1).

Since it is convenient to provide an explicit representation of G_{BMS} we need a suitable coordinate frame on $\Im^+$. Having fixed the triple $(\Im^+, \tilde{h}_B, n_B)$ one is still free to select an arbitrary coordinate frame on the sphere and, using the parameter u of integral curves of n_B to complete the coordinate system, one is free to fix the origin of u depending on $\zeta, \overline{\zeta}$ generally. Taking advantage of stereographic projection one may adopt complex coordinates $(\zeta, \overline{\zeta})$ on the (Riemann) sphere, $\zeta = e^{i\phi} \cot(\vartheta/2)$, ϕ, ϑ being usual spherical coordinates.
Coordinates $(u, \zeta, \overline{\zeta})$ *on* $\Im^+$ *define a* **Bondi frame** *when* $(\zeta, \overline{\zeta}) \in \mathbb{C} \times \mathbb{C}$ *are complex stereographic coordinates on* $\mathbb{S}^2$, $u \in \mathbb{R}$ *(with the origin fixed arbitrarily) is the parameter of the integral curves of* n *and* $(\Im^+, \tilde{h}, n) = (\Im^+, \tilde{h}_B, n_B)$.
In this frame the set G_{BMS} is nothing but $SO(3,1)^\uparrow \times C^\infty(\mathbb{S}^2)$, and $(\Lambda, f) \in$

$SO(3,1)^{\uparrow} \times C^{\infty}(\mathbb{S}^2)$ acts on $\Im^+$ as [13]

$$u \quad \rightarrow \quad u' := K_{\Lambda}(\zeta, \overline{\zeta})(u + f(\zeta, \overline{\zeta}))\,, \tag{3}$$

$$\zeta \quad \rightarrow \quad \zeta' := \Lambda\zeta := \frac{a_{\Lambda}\zeta + b_{\Lambda}}{c_{\Lambda}\zeta + d_{\Lambda}}\,, \qquad \overline{\zeta} \rightarrow \overline{\zeta}' := \Lambda\overline{\zeta} := \frac{\overline{a_{\Lambda}}\overline{\zeta} + \overline{b_{\Lambda}}}{\overline{c_{\Lambda}}\overline{\zeta} + \overline{d_{\Lambda}}}\,. \tag{4}$$

$$K_{\Lambda}(\zeta, \overline{\zeta}) := \frac{(1 + |\zeta|^2)}{|(a_{\Lambda}\zeta + b_{\Lambda})|^2 + + |(c_{\Lambda}\zeta + d_{\Lambda})|^2} \quad \text{and} \quad \left[\begin{array}{cc} a_{\Lambda} & b_{\Lambda} \\ c_{\Lambda} & d_{\Lambda} \end{array}\right] = \Pi^{-1}(\Lambda)\,. \tag{5}$$

Π is the well-known surjective covering homomorphism $SL(2,\mathbb{C}) \rightarrow SO(3,1)^{\uparrow}$. Thus the matrix of coefficients $a_{\Lambda}, b_{\Lambda}, c_{\Lambda}, d_{\Lambda}$ is an arbitrary element of $SL(2,\mathbb{C})$ determined by Λ up to an overall sign. However K_{Λ}[3] and the right hand sides of (4) are manifestly independent from any choice of such a sign.

2.1 Group theoretical data

Starting from (4) and (5), *in a fixed Bondi frame*, G_{BMS} can be viewed as a regular semidirect product between $SO(3,1)^{\uparrow}$, the proper orthocronous subgroup of the Lorentz group and the Abelian additive group $C^{\infty}(\mathbb{S}^2)$ i.e.

$$G_{BMS} = SO(3,1)^{\uparrow} \ltimes C^{\infty}(\mathbb{S}^2).$$

In particular, if $\odot$ denotes the product in G_{BMS}, $\circ$ the composition of functions, $\cdot$ the pointwise product of scalar functions and Λ acts on $(\zeta, \overline{\zeta})$ as said in the right-hand sides of (4):

$$K_{\Lambda'}(\Lambda(\zeta, \overline{\zeta}))K_{\Lambda}(\zeta, \overline{\zeta}) = K_{\Lambda'\Lambda}(\zeta, \overline{\zeta})\,. \tag{6}$$

$$(\Lambda', f') \odot (\Lambda, f) = (\Lambda'\Lambda,\ f + (K_{\Lambda^{-1}} \circ \Lambda) \cdot (f' \circ \Lambda))\,. \tag{7}$$

In the forthcoming discussion concerning the construction of field theories on $\Im^+$, the G_{BMS} group is going to play a key role and, thus, it is necessary to better understand and characterize its structure. To this avail, the first step consists of a carefull analysis of the G_{BMS} subgroups and, in particular, of $C^{\infty}(\mathbb{S}^2)$ whose elements, in the physical literature, are usually referred to as **supertranslations**. As a subgroup it is straightforward to realize that it is an infinite-dimensional abelian ideal of G_{BMS}; thus $C^{\infty}(\mathbb{S}^2)$ as well as the full G_{BMS} group are not ordinary Lie groups. Nonetheless, in the class of infinite-dimensional groups, they lie in a rather privileged class, the nuclear groups first introduced by Gelfand and Vilenkin [24]:

Definition 2.2. A group G is a nuclear Lie group if it exists a neighborhood of the unit element in G which is homeomorphic to a neighborhood of a count-

[3] We adopt the convention of [13] for the analytic expression of $K_{\Lambda}(\zeta, \overline{\zeta})$ which is slightly different from that of [19]. All results from this last cited paper will be adapted accordingly.

ably Hilbert[4] nuclear space.

In order to recognize if the set of supertranslations satisfies in a suitable sense the above definition, we shall make use of a construction for nuclear spaces often used in white noise calculus [6, 25, 26]:

Proposition 2.1. Let $\mathcal{H}$ be any real separable Hilbert space with norm $||,||$ and let A be any self-adjoint densely defined operator on $\mathcal{H}$ such that it exists an orthonormal base $\{e_i\}$ $(i \in \mathbb{N})$ of $\mathcal{H}$ satisfying the conditions:

1. $Ae_i = \lambda_i e_i \quad \forall i \in \mathbb{N}$,
2. $1 < \lambda_1 \leq ... \leq \lambda_n \leq ...$
3. $\exists \alpha \in \mathbb{R}_+$ such that $\sum\limits_{j=1}^{\infty} \lambda_j^{-\alpha} \leq \infty$.

If we introduce for any natural number p the subspace of $\mathcal{H}$

$$\mathcal{E}_p = \left\{ \psi \in \mathcal{H} \quad ||\psi||_p = ||A^p \psi|| < \infty \right\}, \tag{8}$$

we can close each $\mathcal{E}_p$ to an Hilbert space with respect to the norm $||,||_p$ and we can introduce the projective limit space $\mathcal{E} = \bigcap\limits_p \mathcal{E}_p$. Let us equip $\mathcal{E}$ with the projective limit topology τ_p i.e. an open neighborhood of the origin in $\mathcal{E}$ is given by the choice $\epsilon > 0$, $n \in \mathbb{N}$ and by the set $U_{\epsilon,n} = \{\psi \in \mathcal{H}, ||\psi||_n < \epsilon\}$. Then a sequence $\{\psi_m\}_{m \in \mathbb{N}}$ is said to converge to $\psi \in \mathcal{E}$ iff it converges to ψ in every Hilbert space $\mathcal{E}_p$. The pair $(\mathcal{E}, \tau_p)$ is metrizable and complete thus it is a Fréchet space; furthermore the inclusion map $\mathcal{E}_{p+\frac{\alpha}{2}} \hookrightarrow \mathcal{E}_p$ is Hilbert-Schmidt, *i.e.* $\mathcal{E}$ is also a nuclear space.

Theorem 2.1. $C^\infty(\mathbb{S}^2)$[5] is an infinite dimensional nuclear Lie group.

Proof. Let us consider $\mathcal{H} = L^2(\mathbb{S}^2)$ i.e. the space of square integrable functions over the two sphere with respect to the canonical volume element on $\mathbb{S}^2$ and let us take the operator $A = L^2 + kI$ on $\mathcal{H}$ where k is any but fixed real number greater than 1 and where L^2 is the angular momentum operator. Let us remember that the sphere $\mathbb{S}^2$ can be identified with the coset group $\frac{SO(3)}{SO(2)}$. Since L^2 is the second order Casimir operator of $SO(3)$ it coincides with the Laplace Beltrami operator on $\mathbb{S}^2$ up to a factor -2. Thus, with respect to the canonical local chart (θ, φ) on $\mathbb{S}^2$, $2L^2 = -\left(\frac{\partial^2}{\partial\theta^2} + \sin^2\theta \frac{\partial^2}{\partial\varphi^2}\right)$. If we choose the

[4] A topological vector space over $\mathbb{C}$ endowed with a family of inner product norms $\left\{ |\cdot|_p,\ p \in \mathbb{N},\ p \geq 1 \right\}$ is called a *countably Hilbert space* if it is complete with respect to the topology induced by the norms.

[5] From now on, within this paper, $C^\infty(\mathbb{S}^2)$ will actually refer to a set of equivalence classes. A smooth functions $\alpha(\zeta, \overline{\zeta})$ will stand for a representative of the equivalence class $[\alpha(\zeta, \overline{\zeta})]$ where $\alpha(\zeta, \overline{\zeta}) \sim \widetilde{\alpha}(\zeta, \overline{\zeta})$ if they differ for a function of zero measure with respect to the canonical measure over $\mathbb{S}^2$.

basis of spherical harmonics $\{Y_{lm}(\theta,\varphi), l \geq 0, m=-l,...,l\}$, it is immediate to recognize either that A is self-adjoint and densely defined over $\mathcal{H}$ either that

$$AY_{lm}(\theta,\varphi) = \lambda_{lm}Y_{lm}(\theta,\varphi) = [l(l+1)+k]Y_{lm}(\theta,\varphi)$$

Furthermore $1 < \lambda_{00} < \lambda_{1-1} \leq \lambda_{10} < ...$ and it holds that

$$\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\lambda_{lm}^{-\alpha} = \sum_{l=0}^{\infty}(2l+1)[l(l+1)+k]^{-\alpha} < k^{-\alpha} + 4\sum_{l=1}^{\infty} l^{-2\alpha+1},$$

which implies that the first sum is certainly convergent for $\alpha > \frac{1}{2}$.

Thus the hypotheses of the previous proposition are satisfied and we may construct for each $p \in \mathbb{N}$ the space $\mathcal{E}_p$ as in (8) with $A = L^2 + kI$. Furthermore $\mathcal{E}_p \subset \mathcal{E}_q$ for any $p > q \geq 0$ and the inclusion map of $\mathcal{E}_{p+\frac{\alpha}{2}}$ in $\mathcal{E}_p$ is given by the operator $A^{-\frac{\alpha}{2}}$ which is an Hilbert-Schmidt operator with $\left|\left|A^{-\frac{\alpha}{2}}\right|\right|^2_{HS} = \sum\limits_{l=0}^{\infty}(2l+1)\left[l(l+1)+k\right]^{-\alpha}$.

Accordingly, given $\mathcal{E} = \bigcap\limits_p \mathcal{E}_p$ and τ_P, the induced limit topology defined in proposition 2.1, the space $(\mathcal{E}, \tau_p)$ is a nuclear (Fréchet) space. We now need to show that $\mathcal{E}$ coincides with $C^\infty(\mathbb{S}^2)$. Pick any smooth real function $\alpha(\zeta,\overline{\zeta})$; by compactness of $\mathbb{S}^2$, we can immediately conclude that it lies also in $L^2(\mathbb{S}^2) \equiv \mathcal{E}_0$. Furthermore since the operator A is, up to a factor 2, minus the Laplacian operator plus a constant times the identity operator, it maps smooth functions in smooth functions. It implies that $\alpha(\zeta,\overline{\zeta})$ also lies in $\mathcal{E}_p$ for all p and thus also in $\mathcal{E}$. Consequently $C^\infty(\mathbb{S}^2) \subseteq \mathcal{E}$.

The converse is rather more difficult to prove. Referring to $\mathfrak{so}(3)$ as the Lie algebra of $SO(3)$, let us introduce a set of generators $X_i \in \mathfrak{so}(3)$ with $i = 1,...,3$; choosing a representation $\tilde{T}$ of $SO(3)$ in any but fixed Hilbert space $\mathcal{H}$, the corresponding Lie algebra representation T is defined on the set of vectors $u \in \mathcal{H}$ such that it exists the limit

$$\lim_{t\to 0}\frac{1}{t}\left[\tilde{T}\left(\exp(tX)\right) - I\right]u \doteq T(X)u, \tag{9}$$

where $\exp(tX)$ is a one parameter subgroup of elements in $SO(3)$ obtained by means of the exponential mapping. Furthermore, being $SO(3)$ compact, the representation $\tilde{T}$ can be chosen unitary and consequently the operators $iT(X)$ are symmetric.

Fix now $\mathcal{H}$ as $L^2\left(\frac{SO(3)}{SO(2)}\right)$ (where the internal product is defined with respect to the unique $SO(3)$ invariant measure on $\frac{SO(3)}{SO(2)}$) and the unitary representation $T(g)$ as the right action i.e.

$$T(g)\psi(g') = \psi(g'g). \quad \forall g \in SO(3) \ \wedge \ \forall \psi(g) \in L^2\left(\frac{SO(3)}{SO(2)}\right)$$

Choose the generators $X_i \in \mathfrak{so}(3)$ in such a way that the Cartan metric tensor is diagonal; consequently $\Delta = \sum\limits_{i=1}^{3} X_i^2$ is a symmetric elliptic element of the

enveloping algebra of $SO(3)$. Furthermore $T(\Delta) = \sum_{i=1}^{3} T(X_i)^2$ is essentially self adjoint on $L^2\left(\frac{SO(3)}{SO(2)}\right)$ since it coincides with minus the angular momentum/Laplacian operator. Let us now introduce the Garding domain/subspace $D_G \subset \mathcal{H} = L^2(\mathbb{S}^2)$ which is the linear subspace spanned by the linear combinations of

$$T(\varphi)\psi(g') = \int_{SO(3)} d\mu'(g)\varphi(g)T(g)\psi(g'),$$

for all $\varphi(g) \in C^\infty(SO(3))$ and for all $\psi(g') \in L^2\left(\frac{SO(3)}{SO(2)}\right)$. Here $d\mu'(g)$ is the unique Haar measure on $SO(3)$. D_g is dense in $L^2\left(\frac{SO(3)}{SO(2)}\right)$ and it represents a common invariant domain for the generators of the one parameter subgroups of $SO(3)$ i.e. $T(X_i)$ for each i (see theorem 1 ch.11 §1 of [27]). Furthermore $T(\Delta)$ is the Nelson operator which, being an elliptic element in the (right invariant) enveloping algebra of $SO(3)$, is essentially self-adjoint in D_G which, thus, represents a common invariant domain for both $T(\Delta)$ and $T(X_i)$ for all $X_i \in \mathfrak{so}(3)$.

Let us now refer to $\mathcal{L}\left(L^2\left(\frac{SO(3)}{SO(2)}\right)\right)$ as the set of all linear operators on $L^2\left(\frac{SO(3)}{SO(2)}\right)$ and let $\left|\mathcal{L}\left(L^2\left(\frac{SO(3)}{SO(2)}\right)\right)\right|$ be the free abelian semigroup generated by the elements $|A|$ where $|A|$ is the "absolute value"[6] of the operator A. Consequently any element $C \in \left|\mathcal{L}\left(L^2\left(\frac{SO(3)}{SO(2)}\right)\right)\right|$ can be written as the formal sum $C = |\alpha_1|+...+|\alpha_k|$. In particular let us now fix $C = \sum_{i=1}^{3} |T(X_i)|$ and $B = |T(\Delta)|$. According to lemma 7 in ch.11 §3 of [27], it exists a constant $k \leq \infty$ such that, being I the identity operator,

$$C \leq \sqrt{\frac{3}{2}}\,|T(\Delta) - I| \leq \sqrt{\frac{3}{2}}\,[\ |T(\Delta)| + |I|\ ]$$

and, calling $[ad(C)]^n B = \sum_{i_1 \leq ... \leq i_3} |T\,[ad(X_{i_3})...ad(X_{i_1})\Delta]|$ for any $n \in \mathbb{N}$ and for any $i_k = 1,..,3$, it also holds that

$$[ad(C)\,|T(\Delta) - I|\ ]^n \leq k^n\,|T(\Delta) - I| \leq k^n\,[\ |T(\Delta)| + |I|\ ]. \quad \forall n \in \mathbb{N}$$

Thus, $|T(\Delta)|+|I|$ analytically dominates $\sum_{i=1}^{3} |T(X_i)|$. The hypotheses of lemma 5 in ch.11 §1 of [27] are satisfied and we may conclude that, referring to $\overline{T(\Delta)}$ and

[6] We refer to definition of [27] of absolute value of an operator. Pick any triple $A, B, C \in \mathcal{L}\left(L^2\left(\frac{SO(3)}{SO(2)}\right)\right)$. If for all $\psi \in L^2\left(\frac{SO(3)}{SO(2)}\right)$, $||C\psi|| \leq ||A\psi|| + ||B\psi||$, then we represent it as $|C| \leq |A| + |B|$. Thus the absolute value of an operator $A \in \mathcal{L}\left(L^2\left(\frac{SO(3)}{SO(2)}\right)\right)$ is just a symbolic representation for the set consisting of A alone.

$\overline{T(X_i)}$ as the closure respectively of $T(\Delta)$ and $T(X_i)$ (for all i) on $L^2(\mathbb{S}^2)$, the domain of $\overline{T(\Delta)}^n$ is contained in the domain of $\overline{T(X_{i_1})}...\overline{T(X_{i_n})}$ - say $D(X_{i_1}...X_{i_n})$ - for any positive integer n and for any finite sequence $i_1, ..., i_n$. Let now consider the intersection of all the domains $\overline{T(\Delta)}^n$ which, up to the identification of $\left(\frac{SO(3)}{SO(2)}\right)$ with $\mathbb{S}^2$ coincides with $\mathcal{E}$. Then any vector in $\mathcal{E}$ also lies in $D(X_{i_1}...X_{i_n})$ and in particular in the domain of $\overline{T(X)}$ for any $X \in \mathfrak{so}(3)$. By Stone theorem this is the set of elements in $L^2(\mathbb{S}^2)$ for which the limit in (9) exists. Therefore if $\psi \in \mathcal{E}$, $T(g)\psi$ admits all partial derivatives in $g = e$. Let us now introduce the adjoint representation of $SO(3)$ in its Lie algebra which maps $X \in \mathfrak{so}(3)$ into $Ad(g)X = g^{-1}Xg$ for any $g \in \left(\frac{SO(3)}{SO(2)}\right)$. Hence a straightforward application of the properties of a generic representation shows that $\overline{T(X)}T(g)\psi = T(g)\overline{T(Y)}\psi$ where $Y = Ad(g^{-1})X$ for any $g \in SO(3)$. This relation translates in the chain rule $\overline{T(X_1)}...\overline{T(X_k)}T(g)\psi = T(g)\overline{T(Y_1)}...\overline{T(Y_k)}\psi$ where $Y_i = Ad(g^{-1})X_i$. Thus we may conclude that for all $\psi \in \mathcal{E}$ $T(g)\psi$ admits partial derivatives to all order for all $g \in SO(3)$ and thus, acting T as the right multiplication and being the $SO(3)$ action transitive on $\frac{SO(3)}{SO(2)}$, ψ is an infinitely differentiable function over $\frac{SO(3)}{SO(2)}$. As a consequence $\mathcal{E}$ is contained in the space of infinitely differentiable functions on $\mathbb{S}^2$.

We have shown that $C^\infty(\mathbb{S}^2)$ is a nuclear space and, according to definition 2.2, it is also a nuclear Lie group. $\square$

A few remarks are in due course:

Remark 2.1. To each $\mathcal{E}_p$ it is possible to associate the topological dual space $\mathcal{E}'_p$ i.e. the set of continuous linear functional from $\mathcal{E}_p$ to $\mathbb{R}$. It can be closed to Hilbert space with respect to the norm $||,||_{-p}$ such that $||\psi||_{-p} = ||A^{-p}\psi||$ being $||,||$ the $L^2(\mathbb{S}^2)$-norm. We may now define $\mathcal{E}' = \bigcup\limits_p \mathcal{E}_p$ which is the topological dual space of $\mathcal{E}$ and thus we will also refer to it as the space of real distributions on $\mathbb{S}^2$. Consequently we end up with the following **Gelfand triplet**

$$\mathcal{E} \subset L^2(\mathbb{S}^2) \subset \mathcal{E}' \tag{10}$$

together with the set of continuous inclusions $\mathcal{E} \hookrightarrow \mathcal{E}_p \hookrightarrow L^2(\mathbb{S}^2) \hookrightarrow \mathcal{E}'_p \hookrightarrow \mathcal{E}'$. We shall denote with $(,)$ the natural pairing between $\mathcal{E}'$ and $C^\infty(\mathbb{S}^2)$ and it will be subject to the compatibility condition:

$$\big(\alpha(\zeta,\overline{\zeta}), \alpha'(\zeta,\overline{\zeta})\big) = \langle \alpha(\zeta,\overline{\zeta}), \alpha'(\zeta,\overline{\zeta})\rangle_{L^2}, \tag{11}$$

for any $\alpha(\zeta,\overline{\zeta}) \in L^2(\mathbb{S}^2)$ and any $\alpha'(\zeta,\overline{\zeta}) \in C^\infty(\mathbb{S}^2)$. In (11) $\langle,\rangle$ stands for the internal product in $L^2(\mathbb{S}^2)$.

Remark 2.2. The realization of the set of supertranslations as a nuclear space embedded in a Gelfand triplet is completely different from the construction in [19] also followed in [13]. Although the result is ultimately the same, we

have decided to perform a different demonstration for the nuclearity of $C^\infty(\mathbb{S}^2)$ since it will allow us a rigorous construction of the BMS invariant phase space as discussed in the next section. On the opposite McCarthy argument in [19] for nuclearity of the space of supertranslations is a straightforward extension of the demonstration in [28] that $C^\infty([a,b])$ with $[a,b]\subset\mathbb{R}$ is a nuclear space. This is a more common and convenient perspective whether one wants to develop the theory of unitary and irreducible representations for the BMS group by means of Mackey theory of induction.

Remark 2.3. If we adopt the standard topology for $SO(3,1)^\uparrow$ and the product topology for G_{BMS}, then one can straightforwardly conclude that the conditions of definition 2.2 are met, thus the whole BMS group becomes a nuclear Lie group.

To conclude the section we wish to prove a last theorem concerning the abelian ideal of the G_{BMS} group which will be exploited in the discussion of the covariant wave function.

Theorem 2.2. Referring to T^4 as the closed subspace of $C^\infty(\mathbb{S}^2)$ out of the real linear combinations of the first four real spherical harmonics $Y_{lm}(\zeta,\overline{\zeta})$ (with $l=0,1$ and $m=-l,..,l$) and to ST as the closed[7] subspace of $C^\infty(\mathbb{S}^2)$ out of the linear combinations of the real spherical harmonics $\left\{Y_{lm}(\zeta,\overline{\zeta})\right\}_{l>1}$, the following holds:

$$C^\infty(\mathbb{S}^2)=T^4\oplus ST,$$

where $\oplus$ stands for the direct sum.

Proof. The statement of the theorem can be straightforwardly proved in several different ways if we refer to $L^2(\mathbb{S}^2)$. One of the simplest consists of recognizing that the spherical harmonics are an orthonormal complete system of $L^2(\mathbb{S}^2)$ constructed according to standard harmonic functions techniques once $\mathbb{S}^2$ is identified as in the proof of theorem 2.1 with the symmetric space $\frac{SO(3)}{SO(2)}$ (see chapter 10 §3 in [27]). Thus we may claim that $L^2(\mathbb{S}^2)=T^4\oplus ST$ and that, since $C^\infty(\mathbb{S}^2)\subset L^2(\mathbb{S}^2)$, any $\alpha(\zeta,\overline{\zeta})\in C^\infty(\mathbb{S}^2)$ can be univocally decomposed as $\sum\limits_{l=0}^{\infty}\sum\limits_{m=-l}^{l}\alpha_{lm}Y_{lm}(\zeta,\overline{\zeta})$ which converges to $\alpha(\zeta,\overline{\zeta})$ with respect to the topology of $L^2(\mathbb{S}^2)$. Furthermore take into account that, per construction, each $Y_{lm}(\zeta,\overline{\zeta})\in C^\infty(\mathbb{S}^2)$.

We now show that the same sum converges in the topology of $\mathcal{E}\equiv C^\infty(\mathbb{S}^2)$ as constructed in theorem 2.1. Let us thus choose $\epsilon>0$ such that, for any $n\in\mathbb{N}$ greater than a fixed natural number $\bar{n}$, $\left|\left|\alpha(\zeta,\overline{\zeta})-\sum\limits_{l=0}^{n}\sum\limits_{m=-l}^{l}\alpha_{lm}Y_{lm}(\zeta,\overline{\zeta})\right|\right|<\epsilon$, being $||,||$ the L^2-norm. Let us now consider the operator $A=L^2+kI$ $(k>1)$

[7] The closure is here defined for ST, as well as for T^4, with respect to the induced topology τ_p for $C^\infty(\mathbb{S}^2)$.

and let us evaluate

$$\left\|A\left(\alpha(\zeta,\overline{\zeta})-\sum_{l=0}^{n}\sum_{m=-l}^{l}\alpha_{lm}Y_{lm}(\zeta,\overline{\zeta})\right)\right\| \leq$$

$$\leq \left\|L^2\left(\alpha(\zeta,\overline{\zeta})-\sum_{l=0}^{n}\sum_{m=-l}^{l}\alpha_{lm}Y_{lm}(\zeta,\overline{\zeta})\right)\right\| + k\epsilon.$$

Per linearity of L^2 we know that

$$L^2\left(\alpha(\zeta,\overline{\zeta})-\sum_{l=0}^{n}\sum_{m=-l}^{l}\alpha_{lm}Y_{lm}(\zeta,\overline{\zeta})\right) = L^2\alpha(\zeta,\overline{\zeta})+\sum_{l=0}^{n}\sum_{m=-l}^{l}l(l+1)\alpha_{lm}Y_{lm}(\zeta,\overline{\zeta}).$$

We can now exploit again the harmonic function theory according to which if $\alpha'(\zeta,\overline{\zeta}) \in L^2\left(\frac{SO(3)}{SO(2)}\right)$ then the sum

$$\alpha'(\zeta,\overline{\zeta}) = \sum_{l=0}^{\infty}\alpha'_{lm}Y_{lm}(\zeta,\overline{\zeta}),$$
$$\alpha'_{lm} = \int_{\frac{SO(3)}{SO(2)}} d\mu(\zeta,\overline{\zeta})\alpha'(\zeta,\overline{\zeta})Y_{lm}(\zeta,\overline{\zeta})$$

converges in the topology of $L^2\left(\frac{SO(3)}{SO(2)}\right)$ and the decomposition is unique.

Furthermore, since

$$\left[L^2\alpha\right](\zeta,\overline{\zeta}) = \sum_{l=0}^{\infty}\alpha'_{lm}Y_{lm}(\zeta,\overline{\zeta}),$$

we may conclude, by means of the decomposition of $\alpha(\zeta,\overline{\zeta})$ in spherical harmonics, that

$$\alpha'_{lm} = l(l+1)\alpha_{lm}.$$

Consequently

$$\left\|L^2\left(\alpha(\zeta,\overline{\zeta})-\sum_{l=0}^{n}\sum_{m=-l}^{l}\alpha_{lm}Y_{lm}(\zeta,\overline{\zeta})\right)\right\| =$$

$$= \left\|[L^2\alpha(\zeta,\overline{\zeta})]-\sum_{l=0}^{n}\sum_{m=-l}^{l}l(l+1)\alpha_{lm}Y_{lm}(\zeta,\overline{\zeta})\right\| \leq \epsilon,$$

for sufficiently large n; it descends

$$\left\|A\left(\alpha(\zeta,\overline{\zeta})-\sum_{l=0}^{n}\sum_{m=-l}^{l}\alpha_{lm}Y_{lm}(\zeta,\overline{\zeta})\right)\right\| \leq (k+1)\epsilon,$$

i.e. the sum $\sum_{l=0}^{n}\sum_{m=-l}^{l}\left[l(l+1)+k\right]\alpha_{lm}Y_{lm}(\zeta,\overline{\zeta})$ converges to $A\alpha(\zeta,\overline{\zeta})$ in the topology of $L^2(\mathbb{S}^2)$.
The same reasoning leads to the same conclusion with respect to

$$A^p\left[\alpha(\zeta,\overline{\zeta})-\sum_{l=0}^{n}\sum_{m=-l}^{l}\left[l(l+1)+k\right]\alpha_{lm}Y_{lm}(\zeta,\overline{\zeta})\right]$$

for any integer p. Consequently the series $\sum_{l=0}^{n}\sum_{m=-l}^{l}\alpha_{lm}Y_{lm}(\zeta,\overline{\zeta})$ converges to $\alpha(\zeta,\overline{\zeta})$ with respect to $||,||_p$ in each $\mathcal{E}_p$ as introduced in theorem 2.1. Consequently, per definition, the series converges as well to $\alpha(\zeta,\overline{\zeta})$ in $\mathcal{E}\equiv C^\infty(\mathbb{S}^2)$ with respect to the induced topology τ_p.

The uniqueness of the decomposition is accordingly traded from $L^2(\mathbb{S}^2)$ to $C^\infty(\mathbb{S}^2)$ and the hypotheses of closure in $C^\infty(\mathbb{S}^2)$ for ST is justified. $\Box$

Remark 2.4. It is interesting to notice that $\left(C^\infty(\mathbb{S}^2),\tau_p\right)$ constructed as in theorem 2.1 can also be interpreted as a *strong inverse limit of (abelian) Hilbert* Lie (ILH) groups as discussed in [29]. Furthermore, bearing in mind theorem 2.2, the decomposition $C^\infty(\mathbb{S}^2)=T^4\oplus ST$ is a *ILH-splitting* and both T^4 and ST are ILH-subgroups of $C^\infty(\mathbb{S}^2)$. These considerations are automatically traded to the full G_{BMS} group endowed with the product topology and they will be exploited in the forthcoming discussions.

3 BMS free field theory

The development of a field theory invariant under a BMS transformation has been already discussed in previous papers. Nonetheless we shall recast some of the already known results either for sake of completeness either since they play a pivotal role in the next two sections. Thus, as a starting point, we need to review some of the concepts and of the nomenclatures of [10] and, mainly, of [13] *i.e.* we will devout the section to sketch the construction of the BMS unitary and irreducible representations (irreps.) and consequently of the induced wave functions.

3.1 BMS unitary and irreducible representations

As we have demonstrated in the previous section, the G_{BMS} group is an infinite dimensional nuclear Lie group with a semidirect product structure. Thus, in order to develop the theory of irreps. for such a group we shall make use of the inductions techniques as developed by Mackey (see in particular [30, 31] and the recent review in [32]) and extended to a semidirect product with an infinite dimensional abelian ideal by Piard in [33].

As a first step it is much more convenient to replace $SO(3,1)^\uparrow$, the proper ortochronous subgroup of the Lorentz group, with its universal cover $SL(2,\mathbb{C})$.

At a level of theory of representations such operation is not pernicious since, beyond those of G_{BMS}, it introduces only a further irreducible representation induced from the $\mathbb{Z}_2$ subgroup. Thus from now on we will switch from G_{BMS} to $\widetilde{G_{BMS}} = SL(2,\mathbb{C}) \ltimes C^\infty(\mathbb{S}^2)$ which is still a nuclear Lie group.

The second step in Mackey machinery consists on constructing a "character" by means of the following proposition, proved in section 3.2 of [13]:

Proposition 3.1. Given an abelian topological group A, a **character** is a continuous group homomorphism $\chi : A \to U(1)$, the latter being equipped with the natural topology induced by $\mathbb{C}$. If $A = \mathcal{E} \equiv C^\infty(\mathbb{S}^2)$ then it exists a unique real distribution $\beta \in \mathcal{E}'$ such that $\chi\left(\alpha(\zeta,\overline{\zeta})\right) = \exp[i\left(\beta, \alpha(\zeta,\overline{\zeta})\right)]$ for any $\alpha(\zeta,\overline{\zeta}) \in C^\infty(\mathbb{S}^2)$. Here $\left(\beta, \alpha(\zeta,\overline{\zeta})\right)$ stands for the natural dual pairing[8] between $C^\infty(\mathbb{S}^2)$ and its topological dual $\mathcal{E}'$ constructed in remark 2.1.

Remark 3.1. The set of characters $\overline{A}$, equipped with the product operation

$$(\chi_1\chi_2)\left(\alpha(\zeta,\overline{\zeta})\right) = \chi_1\left(\alpha(\zeta,\overline{\zeta})\right)\chi_2\left(\alpha(\zeta,\overline{\zeta})\right) \quad \forall \alpha \in \mathcal{E}$$

is an abelian group called the **dual character group**.
The third step in Mackey's machinery, applied to a regular semidirect product, consists of the identification of three key structures:

Definition 3.1. Consider $G = B \ltimes A$ as the regular semidirect product between a topological abelian group A and any group B. Then for any $\chi \in \overline{A}$, we may associate:

- the **orbit** $\mathcal{O}_\chi \subset \overline{A}$ as the set

  $$\mathcal{O}_\chi = \left\{\chi' \in \overline{A} \mid \exists g \in G \; with \; \chi' = g\chi\right\},$$

  where $g\chi(a) = \chi(g^{-1}a)$ for any $a \in A$ and for any $g \in G$.
- the **isotropy group** $H_\chi \doteq \{g \in G \mid g\chi = \chi\}$.
- the **little group** $L_\chi \subset H_\chi$ as the subset $\{g \in H_\chi \mid g = (\Lambda, 0) \in G\}$.

Remark 3.2. Referring to $\widetilde{G_{BMS}}$ the construction of the structures outlined in definition 3.1 is rather simple since it only requires the identification of the little groups. As shown in [13] and in [19], given a fixed character χ, its isotropy group is $H_\chi = L_\chi \ltimes C^\infty(\mathbb{S}^2)$ whereas the associated orbit $\mathcal{O}_\chi$ is the quotient $\frac{\widetilde{G_{BMS}}}{H_\chi} \sim \frac{SL(2,\mathbb{C})}{L_\chi}$. Furthermore it turns out that all possible little groups L_χ are closed subgroups of $SL(2,\mathbb{C})$ namely $SU(2)$, $SO(2)$, Δ the double cover of the two dimensional Euclidean subgroup, $SL(2,\mathbb{R})$ and the set of all cyclic,

[8] Since $\mathcal{E}'$ is a space of distributions we will also refer to the pairing (β,α) as the evaluation of the distribution β on the test function α.

alternating and dihedral finite dimensional groups of order $n \geq 2$.

Remark 3.3. According to the above definition the orbit $\mathcal{O}_\chi$ should be thought as embedded in the space of characters and as generated by the action of $\frac{SL(2,\mathbb{C})}{L_\chi}$ on χ where L_χ satisfies $L_\chi \chi = \chi$. Nonetheless an equivalent point of view arises exploiting proposition 3.1 according to which $\chi(\alpha) = e^{i\left(\beta, \alpha(\zeta,\overline{\zeta})\right)}$ for a unique choice of $\beta \in \mathcal{E}'$. Thus, since $\Lambda\chi(\alpha(\zeta,\overline{\zeta})) = \chi(\Lambda^{-1}\alpha(\zeta,\overline{\zeta})) = e^{i\left(\Lambda\beta, \alpha(\zeta,\overline{\zeta})\right)}$ for any $\Lambda \in SL(2,\mathbb{C})$, the defining equation for L_χ, i.e. $L_\chi \chi = \chi$, can be traded with an analogous equation in $\mathcal{E}'$ i.e. $L_\chi \bar{\beta} = \bar{\beta}$. Consequently also the orbit $\mathcal{O}_\chi$ in the character space is canonically isomorphic to the orbit embedded in $\mathcal{E}'$ and generated by the action of $\frac{SL(2,\mathbb{C})}{L_\chi}$ on $\bar{\beta}$. For this reason, from now on, we will stick to the much more convenient perspective $\mathcal{O}_\chi \hookrightarrow \mathcal{E}'$ though we will retain the pedex χ for later convenience.

Let us now still focus our attention specifically to $\widetilde{G_{BMS}}$ and let us switch to the more convenient language of fiber bundles. We introduce the *Mackey bundle* $G_\chi = \widetilde{G_{BMS}}\,[\mathcal{O}_\chi, H_\chi, \tau]$ with $\widetilde{G_{BMS}}$ as total space, the orbit of χ as base space, the associated isotropy group H_χ as typical fiber whereas the projection $\tau : G \to \mathcal{O}_\chi$ is suitably chosen case by case[9].

Furthermore, bearing in mind that $\mathcal{O}_\chi$ is $L_\chi \ltimes C^\infty(\mathbb{S}^2)$, we may select a unitary and irreducible representation Σ of $\mathcal{O}_\chi$ acting on a suitably chosen Hilbert space $\mathcal{H}$. Moreover, for any but fixed Σ and for any $g = (\Lambda, \alpha) \in H_\chi$, we may rewrite $\Sigma(g)$ as $\chi(\alpha)\sigma(g)$ where σ is a unitary irrep. of L_χ.

We may proceed constructing the *associated Hilbert bundle* to G_χ as $\mathbb{H} = G_\chi \times_\Sigma \mathcal{H}$ which is a bundle topologically equivalent to the Cartesian product between G_χ and $\mathcal{H}$ whose elements are equivalence classes

$$\begin{aligned}[g,\psi] = \quad & \{(g,\psi) \in G_\chi \times \mathcal{H} \mid (g,\psi) \sim (g',\psi') \\ & iff \quad \exists \tilde{g} \in \mathcal{O}_\chi \mid g' = \tilde{g}g, \wedge \psi' = \Sigma(\tilde{g})\psi\}\,.\end{aligned}$$

$\mathbb{H}$ can be interpreted as a bundle with $\mathcal{O}_\chi$ as the base space, $\mathcal{H}$ as the typical fiber whereas the projection $\tilde{\tau} : \mathbb{H} \to \mathcal{O}_\chi$ maps $[g,\psi]$ in $\tilde{\tau}([g,\psi]) \doteq \tau(g)$.

We are now in position to apply the standard induction technique in order to define a unitary and irreducible representation of the full $\widetilde{G_{BMS}}$-group. Let us thus start introducing the set of smooth sections of the associated Hilbert bundle $\mathbb{H}$ which, up to the choice of a global Borel section $s : \mathcal{O}_\chi \to \widetilde{G_{BMS}}$ for the Mackey bundle G_χ, can be characterized as the set:

$$\Gamma(\mathbb{H})_s = \{\Phi_s : \mathcal{O}_\chi \longrightarrow \mathcal{H} \mid \Phi_s \in C^\infty(\mathcal{O}_\chi, \mathcal{H})\}\,. \tag{12}$$

We may now exploit theorem 3.1 in [13] in order to associate to $\mathcal{O}_\chi$ its unique quasi-invariant measure class $[\mu]$. Thus we may close the space $\Gamma(\mathbb{H})_s$ to an

[9]The existence of τ is not a priori granted in a general scenario but, in the BMS setting, such projection maps have been explicitly identified for all possible little groups [10].

Hilbert space as

$$\widetilde{\mathcal{H}}_{s,\mu} = \left\{ \Phi_s \in C^\infty(\mathcal{O}_\chi, \mathcal{H}) \mid \int\limits_{\mathcal{O}_\chi} d\mu(p) \langle \Phi(p), \Phi(p) \rangle < \infty \right\}, \tag{13}$$

where $p \in \mathcal{O}_\chi$, μ is any representative of $[\mu]$ and $\langle , \rangle$ is the scalar product in $\mathcal{H}$.

Furthermore each element in $\widetilde{\mathcal{H}}_{s,\mu}$ inherits the natural $\widetilde{G_{BMS}}$-action

$$(g\Phi_s)(p) = \sqrt{\frac{d\mu(g^{-1}p)}{d\mu(p)}} (g\Phi_s)(g^{-1}p), \quad \forall g = (\Lambda, \alpha) \in SL(2, \mathbb{C}) \ltimes C^\infty(\mathbb{S}^2) \tag{14}$$

which can be rewritten in the more common and convenient form [30]:

$$(\Lambda\Phi)(p) = \sqrt{\frac{d\mu(\Lambda^{-1}p)}{d\mu(p)}} \sigma \left(s(p)^{-1} \Lambda s(\Lambda^{-1}p) \right) \Phi(\Lambda^{-1}p), \tag{15}$$

$$(\alpha\Phi)(p) = \chi(\alpha)\Phi(p), \tag{16}$$

where $\frac{d\mu(\Lambda^{-1}p)}{d\mu(p)}$ is the Radon-Nikodym derivative and where, bearing in mind remark 3.3, $\chi(\alpha) = e^{i(p,\alpha)}$. Furthermore the following holds:

- applying lemma 1 in §16 of [27], we may conclude that $\widetilde{\mathcal{H}}_{s,\mu}$, whose elements satisfy (14), is isomorphic to the Hilbert space $L^2(\mathcal{O}_\chi, \mu) \otimes \mathcal{H}$.
- applying Mackey's theorem (see [31] or chapter 16 in [27]) (14) is a strongly continuous unitary representation of the $\widetilde{G_{BMS}}$ group induced from $\Sigma = \chi\sigma$.
- all the induced $\widetilde{G_{BMS}}$ unitary representations are irreducibles. Nonetheless a complete list is not available at the moment since all the irreps. must arise either from a transitive $SL(2, \mathbb{C})$-action on $\mathcal{E}'$ or from a cylinder measure with respect to which the $SL(2, \mathbb{C})$-action is strictly ergodic. The latter condition is rather difficult to deal with and the problem of studying it in detail has not been addressed yet.

We may summarize the information from the above discussion in the following statement:

Definition 3.2. We call $\widetilde{G_{BMS}}$ **induced wave function** (or $\widetilde{G_{BMS}}$ free field) any map in (13) which satisfies (14) i.e. it is a square integrable function over $\mathcal{O}_\chi$ with values in a suitably chosen target Hilbert space $\mathcal{H}$ and it transforms under a unitary and irreducible induced representation of the $\widetilde{G_{BMS}}$-group.

In order to complete the analysis of free fields exploiting inducing techniques it is also necessary to construct the full set of Casimir invariants for the unitary

$\widetilde{G_{BMS}}$-representations. Bearing in mind the example of the Poincaré group, one hopes to give a group-theoretical definition to the notion of mass for an induced $\widetilde{G_{BMS}}$-field and to univocally characterize the orbit by the lone value of the invariants.

In order to achieve this goal we exploit the following proposition (see also chapter 4 in [28]):

Proposition 3.2. Given any subspace V of a locally convex linear topological space Ψ any linear continuous functional $\beta : V \to \mathbb{C}$ can be extended to a functional on all Ψ'. Furthermore if we introduce the annihilator of V as:

$$V^0 = \{\beta \in \mathcal{E}' \mid (\beta, v) = 0, \forall v \in V\}, \tag{17}$$

the following holds

1. the factor space $\frac{\Psi'}{V^0}$ is the dual space of V
2. If V is a d-dimensional subspace of Ψ (with $d < \infty$), then also $\frac{\Psi'}{V^0}$ is d-dimensional

Proof. Let us take any continuous linear functional $\beta \in V'$; continuity implies that it exists a neighborhood U of $0 \in \Psi$ such that $|(\beta, \alpha)| \leq 1$ for all $\alpha \in U \cap V$. Let us choose now any absolutely convex neighborhood $U' \subset U$ whose existence is granted by the local convexity of Ψ. We now consider U' as the unit sphere in Ψ of a seminorm such that $||\alpha|| = [sup\,(\lambda)]^{-1}$ where $\lambda\alpha \in U'$ for all $\alpha \in \Psi$. Per construction we end up with $|(\beta, \alpha)| \leq ||\alpha||$ for any $\alpha \in V$. Due to Hahn-Banach theorem (straightforwardly adapted to a space with seminorms), the functional β admits an extension $\bar{\beta}$ on all Ψ which furthermore is bounded i.e. $\left|\left(\bar{\beta}, \alpha\right)\right| \leq ||\alpha||$ for all $\alpha \in \Psi$. Thus it follows that $\left|\left(\bar{\beta}, \alpha\right)\right| \leq 1$ for all $\alpha \in U'$ i.e. $\bar{\beta}$ is also continuous relatively to the topology on Ψ. This concludes the first part of the demonstration.

Let us now consider any $\beta \in \Psi'$ which is also a functional on V being V a subspace of Ψ. Two functionals β_1, β_2 do coincide on V iff they belong to the same coset in $\frac{\Psi'}{V^0}$. Clearly to any $[\beta] \in \frac{\Psi'}{V^0}$ corresponds an element on V' and, if $[\beta_1] \neq [\beta_2]$ on $\frac{\Psi'}{V^0}$, then the corresponding functionals on V' are distinct. The point consists of showing that every linear functional on V' can be constructed in the following way. Let β_0 be any functional in V'. Then, by Hahn-Banach theorem, it can be extended to a linear functional on Ψ' and all the possible extensions coincide on V i.e. they belong to the same coset relatively to V^0. Consequently every linear functional on V corresponds to an element of the factor space $\frac{\Psi'}{V^0}$.

To conclude suppose now that V is finite d-dimensional. Then the above result immediately implies that also V' and consequently $\frac{\Psi'}{V^0}$ is d-dimensional. □

In the $\widetilde{G_{BMS}}$ setting, this proposition can be exploited considering the subspace consisting of the real linear combinations of the first four real spherical

harmonics $Y_{lm}(\zeta,\overline{\zeta}) \subset C^\infty(\mathbb{S}^2)$ with $l=0,1$ and $m=-l,...,l$. This is a four-dimensional subspace which we will refer to as T^4 and which, furthermore, is invariant under the $SL(2,\mathbb{C})$ action induced by (7). Thus, since, according to theorem 2.2, $C^\infty(\mathbb{S}^2)$ is a nuclear space and thus a locally convex linear topological space, we can introduce the projection

$$\pi : \mathcal{E}' \longrightarrow \frac{\mathcal{E}'}{(T^4)^0} \sim \left(T^4\right)', \tag{18}$$

where the isomorphism between $\left(T^4\right)'$ and $\frac{\mathcal{E}'}{(T^4)^0}$ is $SL(2,\mathbb{C})$ invariant. The map (18) enjoys the following remarkable properties whose demonstration is given in [13, 19] (though with slightly different techniques and nomenclatures)

Proposition 3.3. Let $\beta \in \mathcal{E}'$ and let $\{Y^*_{lm}\}$ (with $l=0,1$ and $m=-l,...,l$) be the base of $\left(T^4\right)'$ constructed in such a way that $(Y^*_{lm}, Y_{l'm'}) = \delta_{ll'}\delta_{mm'}$ where $(,)$ refers to the natural pairing between $C^\infty(\mathbb{S}^2)$ and $\mathcal{E}'$. Consider

$$\pi(\beta) = \sum_{l=0}^{1}\sum_{m=-l}^{l} a_{lm} Y^*_{lm},$$

from which we can extract the four vector[10]

$$\widehat{\pi(\beta)}_\mu = -\sqrt{\frac{3}{4\pi}}(a_{00}, a_{1-1}, a_{10}, a_{11}).$$

Moreover, if one defines the real bilinear form B on $\mathcal{E}'$ such that

$$B(\beta_1,\beta_2) = \eta^{\mu\nu}\widehat{\pi(\beta_1)}_\mu \widehat{\pi(\beta_2)}_\nu, \quad \forall \beta_1,\beta_2 \in \mathcal{E}' \tag{19}$$

then B turns out to be $SL(2,\mathbb{C})$ invariant and a Casimir invariant for the $\widetilde{G_{BMS}}$ unitary and irreducible representations.

Remark 3.4. In analogy with the Poincaré counterpart, we will refer to (19) as the defining relation for the $\widetilde{G_{BMS}}$ squared mass m^2. Furthermore such proposition justifies a posteriori the reason for the name of *space of supermomenta* for $\mathcal{E}'$ which is common in the physical literature.

Remark 3.5. In [19] McCarthy showed that the values of m^2 together with the sign of $\widehat{\pi(\beta)}_0$ univocally characterize the orbits only for the little group $SU(2)$. In all other cases to each orbit it is possible to assign a single value of m^2 which does not completely identify/describe it; furthermore there is only one connected subgroup of $SL(2,\mathbb{C})$ which admits $m^2=0$ namely Δ the double

[10] The extraction of a 4-vector from the coefficients of the expansion in dual spherical harmonics a posteriori justifies the symbol T^4 for the subspace of $C^\infty(\mathbb{S}^2)$ generated by $\left\{Y_{lm}(\zeta,\overline{\zeta})\right\}$ with $l=0,1$ and $m=-l,...,l$.

cover of the two dimensional Euclidean subgroup: exactly the same little group associated to the massless fields in a Poincaré invariant theory on Minkowski spacetime.

Though we have fully characterized the full set of $\widetilde{G_{BMS}}$ induced free fields, we need to remember that ultimately our goal is to develop a Lagrangian and an Hamiltonian formulation of $\widetilde{G_{BMS}}$ free field theory. Thus it would be rather prohibitive to deal contemporary with all the possible cases outlined above and we shall make use of a simple but exhaustive "working" example namely the $\widetilde{G_{BMS}}$ scalar field. We distinguish between two cases [13, 19]:

1. **the $\widetilde{G_{BMS}}$ real massive scalar field** which is a map $\Phi \in L^2(\frac{SL(2,\mathbb{C})}{SU(2)_\chi}, \mu)$ whose orbit is generated by the action of $\frac{SL(2,\mathbb{C})}{SU(2)}$ on the real distribution $\bar{\beta} = \sqrt{\frac{4\pi}{3}} m Y^*_{00}$. Furthermore we stress that, since $\bar{\beta} \in \left(T^4\right)'$, we can exploit the $SL(2,\mathbb{C})$ invariant isomorphism on the right hand side of (18) to conclude that the whole orbit is contained in $\left(T^4\right)'$. If we now choose μ as the $SL(2,\mathbb{C})$-invariant measure on the hyperboloid $\frac{SL(2,\mathbb{C})}{SU(2)_\chi}$, then Φ transforms under a $\widetilde{G_{BMS}}$ action as

$$(g\Phi)(\beta) = e^{i\beta(\alpha)}\Phi(\Lambda^{-1}\beta), \quad \forall g = (\Lambda, \alpha) \in \widetilde{G_{BMS}} \ \wedge \ \beta \in \frac{SL(2,\mathbb{C})}{SU(2)}\bar{\beta} \quad (20)$$

2. **the $\widetilde{G_{BMS}}$ real massless scalar field** which is a map $\Phi \in L^2(\frac{SL(2,\mathbb{C})}{\Delta_\chi}, \mu)$ whose orbit is generated by the $\frac{SL(2,\mathbb{C})}{\Delta}$ action on the real distribution $\bar{\beta} = \left(C\delta + K\delta^{(2,2)} + S|z|^{-6}\right)\left(1+|z|^2\right)^3$ where $\delta^{(2,2)}$ represents the derivative of the δ function twice respect to the variable ζ and $\overline{\zeta}$ whereas $K, S \in \mathbb{R}$ and $C \in \mathbb{R} - \{0\}$. Furthermore, as in the massive case, the fixed point $\bar{\beta}$ lies in $\left(T^4\right)'$ and thus we may exploit (18) to conclude that the whole orbit lies in $\left(T^4\right)'$. If we choose μ as an $SL(2,\mathbb{C})$-invariant measure on the light-cone $\frac{SL(2,\mathbb{C})}{\Delta_\chi}$, then Φ transforms under a $\widetilde{G_{BMS}}$ action as

$$(g\Phi)(\beta) = e^{i(\beta,\alpha)}\Phi(\Lambda^{-1}\beta) \quad \forall g = (\Lambda, \alpha) \in \widetilde{G_{BMS}} \ \wedge \ \beta \in \frac{SL(2,\mathbb{C})}{\Delta}\bar{\beta} \quad (21)$$

   Furthermore we shall now remember theorem 3.2 in [13] according to which only the field living on the orbit with $K = S = 0$ coincide with the projection on $\Im^+$ - i. e. null infinity - of a solution for the massless Klein-Gordon equation conformally coupled to gravity in the bulk of any asymptotically flat and globally hyperbolic spacetime. For this reason when we will refer from now on to a real $\widetilde{G_{BMS}}$ massless scalar field we will consider implicitly this physically relevant case. Nonetheless most of results and all the techniques we will make use of may be straightforwardly extended to the general case.

4 The covariant wave function and the associated functional spaces

The aim of this section is to fill a gap in the discussion of field theory at future null infinity as it is appeared up to now in the literature. In [10] and [13] the key ingredient to describe a BMS invariant theory was the so-called canonical or induced point of view according to which a BMS free field is a function transforming under a unitary and irreducible representation of the full BMS group. On the opposite the covariant perspective, though fully equivalent to the canonical one and more common in physics, has not been dealt with in detail. Since this latter point of view is ultimately the most natural one to deal with a Lagrangian or an Hamiltonian formulation of the BMS field theory, we need to amend such lack.

Thus, in order to get over the concept of induced wave function as free field introduced in definition 3.2, the starting point consists on noticing that, for a semidirect product group, the Mackey bundle can be traded with a different one:

$$G' = \widetilde{G_{BMS}}\left[C^\infty(\mathbb{S}^2), SL(2,\mathbb{C}), \tau'\right]. \tag{22}$$

The $\widetilde{G_{BMS}}$ group is still the total space but $\mathcal{E} = C^\infty(\mathbb{S}^2)$ acts as a base space whereas $SL(2,\mathbb{C})$ is the typical fiber and τ' is the natural projection mapping $g = \left(\Lambda, \alpha(\zeta,\overline{\zeta})\right) \in \widetilde{G_{BMS}}$ to $\tau(g) = \alpha(\zeta,\overline{\zeta})$.

We can now exploit either theorem 2.1 either remark 2.1 to introduce the regular semidirect product $SL(2,\mathbb{C}) \ltimes \mathcal{E}'$ with the composition rule between $g = (\Lambda, \beta)$ and $g' = (\Lambda', \beta')$ as

$$g \odot g' = (\Lambda\Lambda', \beta + \Lambda\beta'),$$

where the $SL(2,\mathbb{C})$-action on any element of $\mathcal{E}'$ is

$$\left(\Lambda\beta', \alpha(\zeta,\overline{\zeta})\right) = \left(\beta', \Lambda^{-1}\alpha(\zeta,\overline{\zeta})\right), \quad \forall \alpha(\zeta,\overline{\zeta}) \in C^\infty(\mathbb{S}^2)$$

being $\Lambda^{-1}\alpha(\zeta,\overline{\zeta}) = (K_\Lambda \circ \Lambda^{-1}) \cdot (\alpha(\zeta,\overline{\zeta}) \circ \Lambda^{-1})$ the action of $SL(2,\mathbb{C})$ on a smooth function on $\mathbb{S}^2$ as in (7).
Thus we can introduce

$$\widetilde{G} = \widetilde{G}\left[\mathcal{E}', SL(2,\mathbb{C}), \tilde{\tau}'\right], \tag{23}$$

which is a bundle defined as (22) merely substituting $\mathcal{E}'$ to $C^\infty(\mathbb{S}^2)$. Furthermore, considering both G' and $\widetilde{G}$ as principal bundles and remembering (2.1), we can embed G' in $\widetilde{G}$ by means of the natural homomorphism $i : G' \to \widetilde{G}$ which maps $(\Lambda, \alpha(\zeta,\overline{\zeta})) \in G'$ into the correspondant point in $\widetilde{G}$.

Remark 4.1. From a physical perspective the space $\mathcal{E}'$ is usually referred to as the space of supermomenta since it represents the dual of $C^\infty(\mathbb{S}^2)$, the

space of supertranslations. This nomenclature, originated (see [19, 10]) either in analogy with the Minkowski counterpart where momenta are duals to translations either due to the identification of the $\widetilde{G_{BMS}}$ group with $SL(2,\mathbb{C}) \ltimes L^2(\mathbb{S}^2)$ instead of $SL(2,\mathbb{C}) \ltimes C^\infty(\mathbb{S}^2)$, is rather inconvenient. From one side the enlargement of the abelian ideal of to an Hilbert space, though useful for calculations, is incorrect from the perspective of the holographic principle since, as shown in [13], only within a nuclear topology such as the one associated to $C^\infty(\mathbb{S}^2)$ it is possible to coherently interpret bulk data in terms of boundary ones. From the other side, it allows to identify an isomorphism between the supertranslations and the supermomenta applying Riesz theorem on $L^2(\mathbb{S}^2)$. Such a result does not hold in the generic nuclear scenario and, furthermore, we will show that a function with support on the space of supermomenta - i.e. $\mathcal{E}'$ - cannot never be mapped into a function on the space of supertranslations by means of a Fourier transform. This is in net contrast with the usual paradigm of a field theory in Minkowski spacetime and with the usual physical interpretations of translations and momenta.

The next step consists of following closely the road outlined in the previous subsection; let us thus fix a separable Hilbert space $\mathcal{H}'$ and an $SL(2,\mathbb{C})$ representation ρ acting on it. Then we may construct the associated bundle to $\widetilde{G}$ as

$$\mathbb{H}' = \widetilde{G} \times_{SL(2,\mathbb{C})} \mathcal{H}'$$

which is the set of equivalence classes of points

$$\begin{aligned}[g,\psi] = \ & \Big\{(g,\psi) \in \widetilde{G} \times \mathcal{H}' \mid (g,\psi) \sim (g',\psi') \ iff \\ & \exists \Lambda \in SL(2,\mathbb{C}) \mid g' = \Lambda g \ \wedge \ \psi' = \rho(\Lambda)\psi\Big\}.\end{aligned}$$

A tempting conclusion would now lead to define a new set of wave function as the set of sections for $\mathbb{H}'$ endowed with a suitable regularity conditions. At this stage this is still not possible since, fixing the section $s : \mathcal{E}' \to \widetilde{G}$ such that $\beta \mapsto s(\beta) = (e,\beta)$ being e the identity element in $SL(2,\mathbb{C})$, a section of $\mathbb{H}'$ is a map $\tilde{\Phi} : \mathcal{E}' \to \mathcal{H}'$. Thus we need to introduce a suitable notion of square-integrability on set of functions defined over the topological dual space of a nuclear space. In order to achieve this goal we shall make use of the Minlos theorem (see [6] and in particular [28] for a proof):

Theorem 4.1. [Minlos] Given a real nuclear space V and its topological dual space V', the map $\varphi : V \to \mathbb{C}$ is the characteristic function of the unique probability measure ν on V' such that - calling $(,)$ the pairing between V' and V

$$\varphi(v) = \int_{V'} e^{i(v',v)} d\nu(v') \quad \forall v \in V,$$

iff $\varphi(0) = 1$, φ is continuous on V and positive definite i.e. for any n-tuple of

complex numbers $\{z_i\}_{i=1}^n$ and of elements in V, say $\{v_i\}_{i=1}^n$

$$\sum_{j,k=1}^{n} z_j \bar{z}_k \varphi(v_i - v_k) \geq 0.$$

Lemma 4.1. Fixing the nuclear space $\mathcal{E} = C^\infty(\mathbb{S}^2)$ and its topological dual space $\mathcal{E}'$ along the lines of remark 2.1, the complex valued function

$$\varphi(\alpha(\zeta,\overline{\zeta})) = e^{-\frac{i}{2}\left|\left|\alpha(\zeta,\overline{\zeta})\right|\right|_{L^2}} \tag{24}$$

is the characteristic function of a unique probability measure ν' on $\mathcal{E}'$.

Proof. The demonstration is similar to the standard one for the Schwartz space of real-valued rapidly decreasing test functions on $\mathbb{R}$. As a matter of fact it is straightforward to realize that φ is either continuous either equal to 1 if evaluated in $0 \in C^\infty(\mathbb{S}^2)$.

We need only to verify the positivity of (24). Let us consider any n-tuple of complex numbers $\{z_i\}_{i=1}^n$ and let us call with $C \subset C^\infty(\mathbb{S}^2)$ the subspace (with norm $||,||_{L^2}$) spanned by any but fixed n-tuple of smooth functions over $\mathbb{S}^2$, say $\left\{\alpha_i(\zeta,\overline{\zeta}\right\}_{i=1}^n$. Referring to the standard Gaussian measure on C with μ_C, then any $\alpha(\zeta,\overline{\zeta}) \in C$ satisfies

$$\int_C d\mu_C(\alpha') e^{i\left(\alpha'(\zeta,\overline{\zeta}),\alpha(\zeta,\overline{\zeta})\right)} = e^{i\left|\left|\alpha(\zeta,\overline{\zeta})\right|\right|_{L^2}},$$

where $(,)$ is the internal product in $L^2(\mathbb{S}^2)$. Consequently

$$\sum_{j,k=1}^{n} z_j \bar{z}_k \varphi(\alpha_i(\zeta,\overline{\zeta}) - \alpha_k(\zeta,\overline{\zeta})) = \sum_{j,k=1}^{n} \int_C d\mu_C(\alpha') e^{i\left(\alpha'(\zeta,\overline{\zeta}),\alpha_j(\zeta,\overline{\zeta})-\alpha_k(\zeta,\overline{\zeta})\right)} =$$
$$= \int_C d\mu_C(\alpha') \sum_{j=1}^{n} \left| e^{i\left(\alpha'(\zeta,\overline{\zeta}),\alpha_j(\zeta,\overline{\zeta})\right)}\right|^2 \geq 0,$$

which grants us that φ satisfies the conditions of Minlos theorem. □

The pair $(\mathcal{E}',\nu)$ plays in the BMS field theory the same role that the space of momenta $(\mathbb{R}^4, d^4x)$ plays for a Poincaré invariant field theory over Minkowski spacetime $\mathbb{M}^4$. It it thus natural to ask ourselves if we can define a natural counterpart in the BMS setting also for $L^2(\mathbb{R}^4, d^4x)$ as well for $\mathcal{S}(\mathbb{R}^4)$, the set of rapidly decreasing test functions over $\mathbb{R}^4$ and the space of tempered distributions $\mathcal{S}'(\mathbb{R}^4)$. In order to deal with this question which is fundamental in order to define a covariant BMS (free and interacting) field theory, we still resort to

the powerful techniques of white noise distribution theory [6, 25].

Definition 4.1. We call the *space of square-integrable functions over the supermomenta* the set of equivalence classes of maps

$$L^2(\mathcal{E}',\tilde{\mathcal{H}},\nu)=\left\{\psi:\mathcal{E}'\to\tilde{\mathcal{H}}\ \mid\ \int\limits_{\mathcal{E}'}\langle\psi(\beta),\psi(\beta)\rangle d\nu(\beta)<\infty\right\}, \tag{25}$$

where $\langle,\rangle$ is the internal product on $\tilde{\mathcal{H}}$. Two functions are equivalent if they agree everywhere except in a set of zero measure. This space is also referred[11] to as $\left(L^2\right)_{\tilde{\mathcal{H}}}$.

Eventually we define

Definition 4.2. A $\widetilde{G_{BMS}}$ **covariant field** is a section of the bundle[12] $\mathbb{H}'$ i.e. $\psi\in\left(L^2\right)_{\tilde{\mathcal{H}}}$ which transforms under a unitary representation of the $\widetilde{G_{BMS}}$ group as:

$$\left[U(g)\psi\right](\beta)=e^{i(\beta,\alpha)}D(\Lambda)\psi(\Lambda^{-1}\beta),\quad\forall g=(\Lambda,\alpha(\zeta,\overline{\zeta}))\in\widetilde{G_{BMS}} \tag{26}$$

where $D(\Lambda)$ is a unitary $SL(2,\mathbb{C})$ representation.

As in the induced scenario we shall work with a specific example namely:

Definition 4.3. A $\widetilde{G_{BMS}}$ **real scalar covariant field** is a map ψ which lies in $\left(L^2\right)_{\mathbb{R}}\equiv\left(L^2\right)$ which transforms as:

$$\left[U(g)\psi\right](\beta)=e^{i(\beta,\alpha)}\psi(\Lambda^{-1}\beta).\quad\forall g=\left(\Lambda,\alpha(\zeta,\overline{\zeta})\right)\in\widetilde{G_{BMS}} \tag{27}$$

The definition 4.2 (and consequently 4.3) is at this stage useless until two important aspects are clarified. The first concerns the relation of (26) with the induced wave function (14) which properly characterize a $\widetilde{G_{BMS}}$ free field. Following the seminal work of Wigner for the Poincaré group, such a problem has been dealt with in [10, 13] where it has been shown that both approaches, the induced and the covariant, are equivalent provided that suitable constraints are imposed to (26) in order to reduce it to (14). Nonetheless such constraints should be interpreted as suitable operators acting on $\left(L^2\right)_{\tilde{\mathcal{H}}}$ and their definition requires the introduction of a suitable space of test functions and of generalized functions associated with $\left(L^2\right)_{\tilde{\mathcal{H}}}$. The general theory has been developed in the

[11] In [6, 26] this space is also called "white noise space" though $\mathcal{E}'$, the space of real distributions over $\mathbb{S}^2$ is traded with $\mathcal{S}(\mathbb{R}^d)$ with $d\geq 1$. We feel that, in the BMS setting such nomenclature may be confusing and we will not make use of it.

[12] As in the Poincaré invariant scenario, we implicitly assume that the following continuous global section for the bundle (23) has been chosen namely $s:\mathcal{E}'\to\widetilde{G}$ mapping $\beta\mapsto(I,\beta)$ being I the identity element in $SL(2,\mathbb{C})$.

last twenty years and we refer to [6, 25] for a detailed discussion and for the proofs of the main statements. Conversely we will develop now the construction for the specific scenario we are interested in.

As a starting point and choosing for simplicity $\tilde{\mathcal{H}} = \mathbb{C}$ (or $\mathbb{R}$ by a straightforward adaption of the forthcoming analysis) , we recall that, according to the Itô-Wiener theorem, each function $\psi \in \left(L^2\right)$ can be decomposed as

$$\psi(\beta) = \sum_{n=0}^{\infty} I_n(f_n), \quad f_n \in C^\infty(\mathbb{S}^2)_c^{\hat{\otimes} n} \tag{28}$$

where $C^\infty(\mathbb{S}^2)_c^{\hat{\otimes} n}$ represents the complexification of the n-times symmetric tensor product of $C^\infty(\mathbb{S}^2)$ whereas I_n represents the multiple Wiener integral defined as the linear functional $I_n : C^\infty(\mathbb{S}^2)_c^{\hat{\otimes} n} \to \mathbb{C}$ such that for any $n_1+n_2+... = n$

$$I_n\left(\alpha_1(\zeta,\overline{\zeta})^{\otimes n_1} \hat{\otimes} \alpha_2(\zeta,\overline{\zeta})^{\otimes n_2} \hat{\otimes} ...\right)(\cdot) = \mathcal{F}_{n_1}\left[(\cdot, \alpha_1(\zeta,\overline{\zeta}))\right] \mathcal{F}_{n_2}\left[(\cdot, \alpha_2(\zeta,\overline{\zeta}))\right] ..., \tag{29}$$

where $\mathcal{F}_n\left[x\right] = (-)^n e^{\frac{x^2}{2}} \partial_x^n e^{-\frac{x^2}{2}}$.

A further interesting presentation of an element in $\left(L^2\right)$ consists of showing that the Itô-Wiener decomposition (28) is ultimately equivalent to the following sum (see chapter 5 in [6]):

$$\psi(\beta) = \sum_{n=0}^{\infty} \left(: \beta^{\otimes n} :, f_n\right) \tag{30}$$

where $(,)$ refers to the canonical pairing between $C^\infty(\mathbb{S}^2)$ and $\mathcal{E}'$ whereas $: \beta^{\otimes n} :$ stands for the *Wick tensor*

$$: \beta^{\otimes n} := \sum_{k=0}^{[n/2]} \binom{n}{2k} (2k-1)!! \beta^{\otimes (n-2k)} \widehat{\otimes} \tau^{\otimes k},$$

where $\widehat{\otimes}$ is the symmetrized tensor product and $\tau : \mathcal{E}_c^{\otimes 2} \to \mathbb{C}$, is the trace operator mapping two elements η, ξ in the complexification of $C^\infty(\mathbb{S}^2)$ into

$$(\tau, \eta \otimes \xi) = \langle \eta, \xi \rangle,$$

being $\langle , \rangle$ the internal product in $L^2(\mathbb{S}^2)$.

We may now state the following proposition

Proposition 4.1. Given the densely defined operator on $\left(L^2\right)$ $\Gamma(A)$ such that

$$\Gamma(A)\psi = \sum_{n=0}^{\infty} I_n(A^{\otimes n} f_n),$$

then let us introduce for any $p \in \mathbb{N}$ the set

$$(\mathcal{E})_p = \left\{\psi \in \left(L^2\right) \mid \Gamma(A)^p \psi \in \left(L^2\right)\right\} \tag{31}$$

Closing $(\mathcal{E})_p$ to Hilbert space with respect to the norm $||\psi||_p = ||\Gamma(A)^p\psi||_{(L^2)}$ then we may introduce $(\mathcal{E}) = \bigcap_p (\mathcal{E})_p$ as the projective limit of the sequence $(\mathcal{E})_p$ and $(\mathcal{E}')_p$, $(\mathcal{E}')$ respectively as the topological dual space of $(\mathcal{E})_p$ and of $(\mathcal{E})$. Then $(\mathcal{E})$ is a nuclear space with an associated Gelfand triplet

$$(\mathcal{E}) \subset \left(L^2\right) \subset (\mathcal{E}'),$$

and with the following series of continuous inclusions

$$(\mathcal{E}) \hookrightarrow (\mathcal{E})_p \hookrightarrow \left(L^2\right) \hookrightarrow (\mathcal{E}')_p \hookrightarrow (\mathcal{E}'),$$

where $(\mathcal{E}')_p$ is now the completion of $\left(L^2\right)$ with respect to the norm $||\psi||_{-p} = ||\Gamma(A)^{-p}\psi||_{(L^2)}$. The spaces $(\mathcal{E})$ - endowed with the projective limit topology - and $(\mathcal{E}')$ are respectively called the **space of Hida testing functionals and of Hida distributions**.

The above proposition allows to identify a Gelfand triplet associated to the space $\left(L^2\right)$ and thus we may refer to any element of $(\mathcal{E})$ as a test function and of $(\mathcal{E}')$ as a distribution. We refer to $\langle\langle,\rangle\rangle$ as the natural pairing between $(\mathcal{E})$ and $(\mathcal{E}')$ subjected to the compatibility condition that

$$\langle\langle\psi(\beta), \psi'(\beta)\rangle\rangle = \int_{\mathcal{E}'} d\nu(\beta)\psi(\beta)\psi'(\beta), \tag{32}$$

for any $\psi(\beta) \in (\mathcal{E})$ and for any $\psi'(\beta) \in \left(L^2\right)$. Nonetheless, in order to correctly identify the constraints which reduce the covariant to the induced wave function, we need now to introduce the concepts of *multiplication operator*.

Definition 4.4. Given any $\alpha(\zeta,\overline{\zeta}) \in C^\infty(\mathbb{S}^2)$, we call **multiplication operator** (along the α-direction) the continuous operator $Q_\alpha : (\mathcal{E}) \to (\mathcal{E})$ such that

$$Q_\alpha\varphi(\beta) = (\beta, \alpha(\zeta,\overline{\zeta}))\varphi(\beta), \quad \forall\varphi \in (\mathcal{E}) \wedge \forall\alpha(\zeta,\overline{\zeta}) \in C^\infty(\mathbb{S}^2) \tag{33}$$

Furthermore we refer to $\widetilde{Q}_\alpha$ as the continuous extension of Q_α to $(\mathcal{E}')$ which is defined in analogy with (33).

Bearing in mind the above definitions we are now facing the following situation: a $\widetilde{G_{BMS}}$ covariant field (scalar or not) is, according to its definition and to proposition 4.1, a square integrable function over the space of distributions over $\mathbb{S}^2$ or, as well, a Hida testing functional if we take into account that $(\mathcal{E}) \subset \left(L^2\right) \subset (\mathcal{E}')$.

At the same time a $\widetilde{G_{BMS}}$ free field is defined as in (13) i.e. it is a square integrable function whose support is a finite dimensional homogeneous space $\mathcal{O}_\chi$ embedded in $\mathcal{E}'$.

We underline again that, according to Wigner seminal work for the Poincaré scenario, the above two points of view are equivalent provided that suitable

constraints are imposed on the covariant field in order to reduce it to the induced counterpart. In the BMS setting the overall idea is the same though we face a substantial difference since, as we have outlined above, the covariant field has support on a functional space and thus it is apparently rather counterintuitive that, starting from a field $\psi \in L^2(\mathcal{E}', \nu) \otimes \mathcal{H}$ we shall find a constraint reducing it to a function $\Psi \in L^2(\mathcal{O}_\chi, \nu) \otimes \mathcal{H}'$. We have already addressed this problem in [13] though not in the rigorous frame of Hida distributions. We will now provide a constructive demonstration of Wigner idea for the specific scenario of a real scalar field with mass m i.e. $\mathcal{H} = \mathcal{H}' = \mathbb{R}$ and the orbit of the induced wave function is the hyperboloid $\frac{SL(2,\mathbb{C})}{SU(2)}$ if $m^2 > 0$ or $\frac{SL(2,\mathbb{C})}{\Delta}$ if $m^2 = 0$.

The starting point consists of introducing a finite dimensional counterpart of the elements in $(\mathcal{E})'$. We will state now some results first appeared in [34] and here stated in our specific scenario. Adaption to the general scenario is straightforward.

Definition 4.5. Let $C^\infty(\mathbb{S}^2) \subset \mathcal{H} \subset \mathcal{E}'$ be the Gelfand triplet constructed in remark 2.1 out of which the space of Hida distributions $(\mathcal{E}')$ has been constructed as in proposition 4.1. Then if we choose any k-tuple $\{e_1, ...e_k\} \subset L^2(\mathbb{S}^2)$ with $k < \infty$ and if we refer to V as the real linear space spanned by $e_1, ..., e_k$, we may introduce the space $(\mathcal{E}')_V$ as the $(\mathcal{E}')$-closure of all polynomials in $\langle \cdot, \vec{e} \rangle \doteq (\langle \cdot, e_1 \rangle, ..., \langle \cdot, e_k \rangle)$. Then we call $\psi \in (\mathcal{E}')$ a **finite dimensional Hida distribution** if $\psi \in (\mathcal{E}')_V$ for some finite dimensional subspace V constructed as above. We call $(\mathcal{E})_V \doteq (\mathcal{E}) \cap (\mathcal{E}')_V$ the space of **finite dimensional Hida test functions**.

The above definition clearly underlines that certain specific Hida distributions/testing functionals could be interpreted as finite dimensional distributions/testing functionals. The natural subsequent step would be to interpret them as Schwartzian generalized functions or testing functionals over $\mathbb{R}^k$ though it is rather straightforward to realize that the Gelfand triplet $\mathcal{S}(\mathbb{R}^k) \subset L^2(\mathbb{R}^k) \subset \mathcal{S}'(\mathbb{R}^k)$ does not fit in this picture since a priori there is no reason why a finite dimensional Hida distribution should lie in the dual space of rapidly decreasing test functions. Thus we need to introduce a new auxiliary Gelfand triplet; the starting point consists in $\mathcal{P}(\mathbb{R}^k)$ which is the space of polynomials in $x_\mu = (x_1, ..., x_k) \in \mathbb{R}^k$ with $k < \infty$. Referring to μ_k as the standard Gaussian measure on $\mathbb{R}^k$, we may close $\mathcal{P}(\mathbb{R}^k)$ to Hilbert space - say $\overline{\mathcal{P}(\mathbb{R}^k)}$ - with respect to the inner product

$$(F, G) = \int\limits_{\mathbb{R}^k} F(x_\mu) G(x_\mu) d\mu_k(x_\mu). \quad \forall F, G \in \mathcal{P}(\mathbb{R}^k)$$

We shall construct a Gelfand triplet out of this Hilbert space considering the Ornstein-Uhlenbeck operator on $\mathbb{R}^k$ i.e. $L = \nabla - \sum\limits_{i=1}^{k} x_i \frac{\partial}{\partial x_i}$, where x_i are the Cartesian coordinates on $\mathbb{R}^k$ whereas ∇ is the Laplacian operator on $\mathbb{R}^k$. As shown in [34] we can now exploit proposition (2.1) with respect to the basis

for L in $\overline{\mathcal{P}(\mathbb{R}^k)}$ given by the vector $H_{\mathbf{n}}(x_\mu) = \prod_{i=1}^{k} H_{\mathbf{n_i}}(x_i)$ being $H_{\mathbf{n_i}}$ the n_i-th Hermite polynomial. Thus we introduce the sequence of spaces - with respect to the parameter $t \in \mathbb{R}$ -

$$\mathcal{I}_t(\mathbb{R}^k) = \left\{ \psi \in \overline{\mathcal{P}(\mathbb{R}^k)} \;\mid\; \exp(-tL)\psi \in \overline{\mathcal{P}(\mathbb{R}^k)} \right\}. \tag{34}$$

Closing this space to Hilbert space with respect to the internal product

$$(F, G)_t = (\exp(-tL)F, \exp(-tL)G), \quad \forall F, G \in \mathcal{I}_t(\mathbb{R}^k),$$

one ends up for any real positive value of t with the sequence of continuous inclusions

$$\mathcal{I}_t(\mathbb{R}^k) \hookrightarrow \overline{\mathcal{P}(\mathbb{R}^k)} \hookrightarrow \mathcal{I}_{-t}(\mathbb{R}^k).$$

Considering now the projective limit space $\mathcal{I}(\mathbb{R}^k) = \bigcap_t \mathcal{I}_t(\mathbb{R}^k)$, endowed as in proposition 2.1 with the projective limit topology, we may construct the new Gelfand triplet

$$\mathcal{I}(\mathbb{R}^k) \subset \overline{\mathcal{P}(\mathbb{R}^k)} \subset \mathcal{I}'(\mathbb{R}^k), \tag{35}$$

where $\mathcal{I}'(\mathbb{R}^k)$ is the topological dual space of $\mathcal{I}(\mathbb{R}^k)$.

We can now formulate a characterization theorem for finite dimensional Hida testing functionals whose demonstration has been given in [34] and which is here stated in terms of our specific framework:

Theorem 4.2. Referring to the Gelfand triplet $C^\infty(\mathbb{S}^2) \subset L^2(\mathbb{S}^2) \subset \mathcal{E}'$ let us choose, as in definition 4.5, a k-tuple $\{e_i\}_{i=1}^k \in C^\infty(\mathbb{S}^2)$ whose elements are mutually orthogonal with respect to the inner product in $L^2(\mathbb{S}^2)$ and let us call $V = span\,\{e_i\}_{i=1}^k$. Then for any finite dimensional Hida testing functional $\psi \in (\mathcal{E})_V$, it exists a function $F \in \mathcal{I}(\mathbb{R}^k)$ such that $\varphi = F(x_\mu)$ where $x_\mu = (\langle \cdot, e_1 \rangle, ..., \langle \cdot, e_k \rangle)$.

Furthermore let us introduce the projector $\pi_V : \mathcal{E}' \to V$ which maps $y \in \mathcal{E}'$ to $\pi_V(y) = \sum_{i=1}^{k} (y, e_i)\, e_i$ where $(,)$ stands for the pairing between $C^\infty(\mathbb{S}^2)$ and $\mathcal{E}'$. Then π_V automatically induces a projection operator $\Pi_V : (\mathcal{E}') \to (\mathcal{E}')_V$ such that $\Pi_V \doteq \Gamma(\pi_V)$ maps any $\psi \in (\mathcal{E}')$ in

$$\Pi_V \psi = \Pi_V \left(\sum_{n=0}^{\infty} I_n(f_n) \right) = \sum_{n=0}^{\infty} I_n(\pi_v^{\otimes n} f_n), \tag{36}$$

being I_n the multiple Wiener integral as in (28). Thus we conclude that $\psi \in (\mathcal{E}')_V$ iff

$$\Pi_V \psi = \psi. \tag{37}$$

The above theorem grants us that any Hida testing functional which satisfies (37) naturally identifies a function lying in $\mathcal{I}(\mathbb{R}^k)$; this is not the answer we were

looking for since we ultimately seek an element at least in $\mathcal{S}(\mathbb{R}^k) \subset L^2(\mathbb{R}^k)$. Thus we need to exploit another theorem proved in [34] and here adapted to our specific scenario:

Proposition 4.2. If a function $F(x_\mu)$ lies in $\mathcal{I}(\mathbb{R}^k)$ then $F(x_\mu)e^{-\frac{1}{4}\delta^{\mu\nu}x_\mu x_\nu}$ lies in $\mathcal{S}(\mathbb{R}^k)$ being $\delta^{\mu\nu}$ the Kroneker delta.

We have now all the ingredient to exploit the theory of finite-dimensional Hida distributions to construct the equations of motion for the BMS free field. The first step consists of remembering that both the orbit of the massive and massless real scalar $\widetilde{G_{BMS}}$ field lies in $\left(T^4\right)'$. Bearing in mind that such a space is generated by real linear combinations out of the basis $\{Y^*_{lm}\}^1_{l=0} \subset \mathcal{E}'$ defined as $\left(Y^*_{lm}, Y_{l'm'}(\zeta,\overline{\zeta})\right) = \delta_{ll'}\delta_{mm'}$, it is natural to choose $V = T^4 = \left\{Y_{lm}(\zeta,\overline{\zeta})\right\}_{l=0,1}$.

Lemma 4.2. The orbit/support of a covariant real (massive or massless) scalar field ψ lies in $(T^4)'$ iff

$$\Pi_{T^4}\psi(\beta) = \psi(\beta). \tag{38}$$

Proof. We exploit the Itô-Wiener decomposition of a generic functional in $\left(L^2\right)$ as $\psi(\beta) = \sum\limits_{n=0}^{\infty} (:\beta^{\otimes n}:, f_n)$ and (36). According to this latter equation and introducing $e_\mu = \left(Y_{00}(\zeta,\overline{\zeta}), ..., Y_{11}(\zeta,\overline{\zeta})\right)$, (38) reads:

$$\sum_{n=0}^{\infty}\left(:\beta^{\otimes n}:, \sum_{\mu=0}^{4}\left(e_\mu, f_n\right)e_\mu\right) = \sum_{n=0}^{\infty}\left(:\beta^{\otimes n}:, f_n\right),$$

which is satisfied iff $\beta \in \left(T^4\right)'$ or f_n lies in $\left(T^4\right)^{\widehat{\otimes} n}$ for any n. In this latter case we shall make use of proposition 3.2 - more precisely of the considerations in its proof - to conclude that, whenever a generic distribution $\beta \in \mathcal{E}'$ is evaluated with a test function $\alpha(\zeta,\overline{\zeta}) \in T^4$, this is equal to extract from β a representative in an equivalence class of $\frac{\mathcal{E}'}{(T^4)^0}$ and evaluate it with $\alpha(\zeta,\overline{\zeta})$. Bearing now in mind the $SL(2,\mathbb{C})$-invariant isomorphism between $\frac{\mathcal{E}'}{(T^4)^0}$ with $\left(T^4\right)'$, the statement of the theorem is naturally implied. $\Box$

For later convenience it is interesting to notice at this stage that the above equation of motion can be also written in terms of operators acting on the covariant wave function namely, referring to definition 4.4, the following lemma holds:

Lemma 4.3. Bearing in mind the decomposition in theorem 2.2, a field $\psi \in \left(L^2\right)$ satisfies (38) iff

$$Q_{\alpha(\zeta,\overline{\zeta})}\psi(\beta) = 0. \quad \forall\alpha(\zeta,\overline{\zeta}) \in ST \tag{39}$$

Proof. According to definition 4.4, $Q_\alpha(\zeta,\overline{\zeta})\psi(\beta) = \left(\beta, \alpha(\zeta,\overline{\zeta})\right)\psi(\beta)$; it is immediate to realize that if (38) holds, then lemma 4.2 grants us that β can be chosen in $\left(T^4\right)'$ and, unless $\psi(\beta)$ is identically vanishing, (39) is zero iff $\left(T^4\right)' \subseteq (ST)^0$ which is the annihilator of ST. At the same time if we suppose that (39) holds then $\beta \in (ST)^0$ and (38) holds iff $(ST)^0 \subseteq \left(T^4\right)'$. We need only to demonstrate that it exists an isomorphism between $\left(T^4\right)'$ and $(ST)^0$.

The starting point consists of exploiting theorem 2.2 according to which the factor space $\frac{C^\infty(\mathbb{S}^2)}{ST}$ is isomorphic to the subspace $T^4 \subset C^\infty(\mathbb{S}^2)$. Accordingly, per duality, also $\left(T^4\right)'$ is isomorphic to $\left(\frac{C^\infty(\mathbb{S}^2)}{ST}\right)'$. Furthermore any $\beta \in \left(T^4\right)'$ can be extended according to theorem 3.2 to a functional $\tilde{\beta}$ on $\mathcal{E}'$ in such a way that, given any two $\alpha(\zeta,\overline{\zeta}), \alpha'(\zeta,\overline{\zeta}) \in C^\infty(\mathbb{S}^2)$, $\tilde{\beta}\left(\alpha(\zeta,\overline{\zeta})\right) = \tilde{\beta}\left(\alpha'(\zeta,\overline{\zeta})\right)$ if $\alpha(\zeta,\overline{\zeta}) - \alpha'(\zeta,\overline{\zeta}) \in ST$. Per linearity of the elements in $\mathcal{E}'$, it implies $\tilde{\beta}\left(\alpha(\zeta,\overline{\zeta}) - \alpha'(\zeta,\overline{\zeta})\right)$ vanishes i.e. $\tilde{\beta} \in (ST)^0$ and $\left(T^4\right)' \subseteq (ST)^0$.

To show the opposite inclusion let us start from any $\beta \in (ST)^0$. We can now exploit theorem 2.2 according to which ST is a subspace of $C^\infty(\mathbb{S}^2)$ and thus, according to theorem 3.2, β can be extended to a functional in $\mathcal{E}'$. Choose any such extension - say $\tilde{\beta}$ - and evaluate it on any $\alpha(\zeta,\overline{\zeta}) \in C^\infty(\mathbb{S}^2)$. Still according to theorem 2.2, $\alpha(\zeta,\overline{\zeta})$ can be univocally split in the sum of $\alpha'(\zeta,\overline{\zeta}) \in T^4$ and $\tilde{\alpha}(\zeta,\overline{\zeta}) \in ST$. Thus, per linearity,

$$\tilde{\beta}(\alpha(\zeta,\overline{\zeta})) = \tilde{\beta}(\alpha'(\zeta,\overline{\zeta})) + \tilde{\beta}(\tilde{\alpha}(\zeta,\overline{\zeta})) = \tilde{\beta}(\alpha'(\zeta,\overline{\zeta})),$$

where the last equality holds since $\tilde{\beta}$ must agree with β on ST. Thus the above equation grants us that $\tilde{\beta} \in \left(T^4\right)'$ i.e. $(ST)^0 \subseteq \left(T^4\right)'$, which concludes the demonstration. □

We have now identified the class of covariant $\widetilde{G_{BMS}}$ scalar fields ψ which are supported on $\left(T^4\right)'$. The last step consists of choosing suitable constraints which grant us that ψ is supported either on the hyperboloid $\frac{SL(2,\mathbb{C})}{SU(2)}$ either on the light cone $\frac{SL(2,\mathbb{C})}{\Delta}$. From a physical perspective this amounts to assign a fixed value for the mass to the covariant field and, from an operative point of view, it translates in the following lemma:

Lemma 4.4. A $\widetilde{G_{BMS}}$ covariant scalar field ψ has support on the orbit generated by $\frac{SL(2,\mathbb{C})}{SU(2)}$ action on $\bar{\beta}_1 = \sqrt{\frac{3}{4\pi}}mY^*_{00}$ or on that generated by $\frac{SL(2,\mathbb{C})}{\Delta}$ action on $\bar{\beta}_2 = C\delta$ iff, besides (38), ψ satisfies

$$\eta^{\mu\nu}Q_{e_\mu}Q_{e_\nu}\psi(\beta) = \begin{cases} 0 & \text{for the little group } \Delta \\ m^2\psi(\beta) & \text{for the little group } SU(2) \end{cases}, \qquad (40)$$

where $e_\mu = \left(Y_{00}(\zeta,\overline{\zeta}), ..., Y_{11}(\zeta,\overline{\zeta})\right)$ is the 4-vector of elements in $C^\infty(\mathbb{S}^2)$ and $\eta^{\mu\nu} = diag(-1,1,1,1)$.

Proof. Suppose that $\psi(\beta)$ is supported on the orbit generated by $\frac{SL(2,\mathbb{C})}{L_i}$ on $\bar{\beta}_i$ where $i = 1, 2$ and $L_1 = SU(2)$ and $L_2 = \Delta$. Then, for any point β on one of the two orbit, it exists $\Lambda \in SL(2,\mathbb{C})$ such that $\beta = \Lambda\bar{\beta}_i$ and the following chain of identities holds:

$$\eta^{\mu\nu} Q_{e_\mu} Q_{e_\nu} \psi(\beta) = \eta^{\mu\nu} Q_{e_\mu} Q_{e_\nu} \psi(\Lambda\bar{\beta}_i) = \eta^{\mu\nu} \left(e_\mu, \Lambda\bar{\beta}_i\right) \left(e_\nu, \Lambda\bar{\beta}_i\right) \psi(\Lambda\bar{\beta}_i) =$$

$$= B(\Lambda\bar{\beta}_i, \Lambda\bar{\beta}_i)\psi(\Lambda\bar{\beta}_i) = m^2\psi(\beta),$$

where m^2 is either 0 or different from 0 depending on the chosen little group. In the above chain of identities we have exploited the multiplication operator as introduced in definition 4.4 whereas, in the last two identities, we refer to proposition 3.3 and, in particular, to the definition of the real bilinear form (19) and its $SL(2,\mathbb{C})$ invariance.

The converse is rather straightforward. Suppose a covariant scalar $\widetilde{G_{BMS}}$ field satisfies (38). Then $\beta \in \left(T^4\right)'$ and (40) becomes

$$\left[\eta^{\mu\nu}(\beta, e_\mu)(\beta, e_\nu) - m^2\right] \psi(\beta) = 0. \quad \beta \in \left(T^4\right)'$$

Thus, unless $\psi(\beta)$ is identically vanishing, $\eta^{\mu\nu}(\beta, e_\mu)(\beta, e_\nu) - m^2 = 0$ which is, depending on the chosen value for m^2, the defining equation for the mass hyperboloid or for the light cone realized in $\mathbb{R}^4$. We need at last to show that the orbit is necessarily generated by the fixed point $\bar{\beta}_i$. This is still straightforward; suppose that $m^2 \neq 0$, then we just need to exploit that $\eta^{\mu\nu}(\beta, e_\mu)(\beta, e_\nu)$ is the $SL(2,\mathbb{C})$ invariant bilinear form $B(\beta,\beta)$ as in (19). Thus we may find $\Lambda \in SL(2,\mathbb{C})$ such that $B(\beta,\beta) = B(\Lambda\beta, \Lambda\beta) = m^2$ and $B(\Lambda\beta, \Lambda\beta) = (\beta', e_0)(\beta', e_0)$ where $\beta' = \Lambda\beta$. Since $e_0 = Y_{00}(\zeta, \bar{\zeta})$, β' should be equal to a constant times Y^*_{00} plus a term lying in the annihilator[13] of e_0 - say $(Y_{00})^0$. To be rigorous one now should exploit proposition 3.2 to show that it exists an isomorphism between $\frac{(T^4)'}{(Y_{00})^0}$ and the space dual to one dimensional subspace of T^4 generated by Y_{00}. Thus one can always choose the representative in such factor group in such a way that it coincides with $\bar{\beta}_1$ i.e. the distribution generating the orbit for the massive canonical $\widetilde{G_{BMS}}$ scalar field. An identical procedure leads to the same conclusion for the massless case and thus the statement is proved. □

We have almost completed our task. According to lemma 4.2 and 4.4 we have shown that a $\widetilde{G_{BMS}}$ covariant scalar field $\psi \in \left(L^2\right)$ satisfying (27) can be reduced to a function on the mass hyperboloid or on the light cone transforming under a scalar $\widetilde{G_{BMS}}$ unitary and irreducible representation (respectively induced from the $SU(2)$ and the Δ subgroups of $SL(2,\mathbb{C})$) iff it satisfies the equations (38) and (40).

The tricky point is the following: can we conclude that this function is square integrable with respect to the measure on each orbit? At this stage

[13] The reader should bear in mind that the annihilator of any $e_\mu \in T^4$ is the set of elements f in $\left(T^4\right)'$ such that $f(ke_\mu) = 0$ for any $k \in \mathbb{R}$.

this is definitely not possible since theorem 4.2 grants us that a map $\psi \in \left(L^2\right)$ such that $\Pi_{T^4}\psi(\beta) = \psi(\beta)$ is in one to one correspondence with the functions in $\mathcal{I}(\mathbb{R}^4)$ - say $\psi(p_\mu)$ with $p_\mu =< \beta, e_\mu >$ - which is is continuously embedded in the Hilbert space $\overline{\mathcal{P}(\mathbb{R}^4)}$ which we remember being the Hilbert space of polynomial function with respect to the canonical Gaussian measure on $\mathbb{R}^4$. Thus, from one side this inclusion justifies the claim that the covariant field satisfying (38) and (40) transforms under a unitary $\widetilde{G_{BMS}}$ induced representation whereas from the other side it allow us to exploit theorem 4.2 to claim that $\tilde{\psi}(p_\mu) = e^{-\delta^{\mu\nu}p_\mu p_\nu}\psi(p_\mu)$ lies in $\mathcal{S}(\mathbb{R}^4)$ i.e. $\tilde{\psi}(p_\mu)$ is square-integrable with respect to the Lesbegue measure on $\mathbb{R}^4$. The remaining constraint (40) does not harm the previous reasoning since it corresponds in the space $\mathcal{I}(\mathbb{R}^4)$ to impose the usual equation

$$\left[\eta^{\mu\nu}p_\mu p_\nu - m^2\right]\psi(p_\mu) = 0,$$

which is also identically satisfied by $\tilde{\psi}(p_\mu) = 0$. Thus we can summarize the full construction in the following theorem:

Theorem 4.3. A covariant $\widetilde{G_{BMS}}$ (massive or massless) scalar field $\psi : \mathcal{E}' \to \mathbb{R}$ which transforms as (27) and which satisfies the equations (38) and (40) corresponds to a $\widetilde{G_{BMS}}$ induced scalar field ((20) or (21)) up to the rescaling of the latter by $e^{-\delta^{\mu\nu}p_\mu p_\nu}$.

Remark 4.2. The construction outlined above refers to the special case of the scalar fields. In particular, for the massless case, we referred to an induced wave function living on a rather specific orbit. The real purpose behind such a choice arises from a physical perspective since, as we have outlined before, up to now the $\widetilde{G_{BMS}}$ fields on null infinity which can be physically interpreted from an holographic point of view are those supported on $\left(T^4\right)'$ (see for example [7, 13]). Nonetheless the overall idea for the above construction can be slavishly applied to a generic $\widetilde{G_{BMS}}$ covariant field in order to reduce it to its induced counterpart. The real tricky issue would be to construct case by case the suitable constraints and in particular to select a specific set of orthonormal functions in $L^2(\mathbb{S}^2)$ out of which construct a finite dimensional Hida testing functional starting from the whole covariant field.

4.1 $\widetilde{G_{BMS}}$ equations of motion as evolution equations

To conclude this section it is natural to deal with the following remark: the equations of motion for a $\widetilde{G_{BMS}}$ (massless or massive) scalar field are constraint equations in direct analogy with the counterpart in a Poincaré invariant free field in the momenta space. Nonetheless it is often more convenient to deal either in classical either in quantum field theory with an evolution problem i.e. (at least) a partial differential equation. In order to switch to this perspective, in our scenario, we need to introduce two key ingredients: a differential operator on the space of Hida testing functionals and distributions and a suitable notion

of "Fourier-like" transform $\mathcal{F}$. The answer to this query has been developed and extensively discussed in [6, 25] and we will limit ourselves to the main definitions:

Definition 4.6. Let us consider any Hida testing functional $\psi(\beta)$ on $(\mathcal{E})$; we define the Gateaux derivative of $\psi(\beta)$ along the direction $\tilde{\beta} \in \mathcal{E}'$ as the continuous operator $\mathcal{D}_{\tilde{\beta}} : (\mathcal{E}) \to (\mathcal{E})$ such that

$$\mathcal{D}_{\tilde{\beta}}\psi(\beta) = \lim_{\epsilon \to 0} \frac{\psi(\beta + \epsilon\tilde{\beta}) - \psi(\beta)}{\epsilon} = \sum_{n=1}^{\infty} \left(: \beta^{\otimes(n-1)} :, (\tilde{\beta}, f_n) \right) \tag{41}$$

The operator $\mathcal{D}_{\tilde{\beta}}$ admits a unique continuous extension to an operator $\widetilde{\mathcal{D}}_{\tilde{\beta}} : (\mathcal{E}') \to (\mathcal{E}')$.

We are going to state now an important result which relates the multiplication operator with the Gateaux derivative. The following lemma, proved in [6], also shows that, opposite to the usual behaviour such as on the space of Schwartz test functions, the multiplication operators is a sort of "derivative operator" i.e. it obeys a Liebnitz rule.

Lemma 4.5. For any $\alpha(\zeta, \overline{\zeta}) \in C^{\infty}(\mathbb{S}^2)$, the following equality holds:

$$Q_{\alpha(\zeta,\overline{\zeta})} = \mathcal{D}_{\alpha(\zeta,\overline{\zeta})} + \mathcal{D}^*_{\alpha(\zeta,\overline{\zeta})},$$

which is meant as a continuous operator form $(\mathcal{E})$ into itself. Furthermore, for any $\beta \in \mathcal{E}'$, it also holds:

$$Q_{\beta} = \mathcal{D}_{\beta} + \mathcal{D}^*_{\beta},$$

which is meant as a continuous operator from $(\mathcal{E})$ into $(\mathcal{E}')$.

In order to switch from a constraint equation such as (39) and (40) to an evolution equation, the natural step in the canonical formulation of quantum field theory over Minkowski background consists of performing a Fourier transform $\tilde{\mathcal{F}}$. In this latter framework such a transformation is a continuous linear operator from the space of Schwartz test function $\mathcal{S}(\mathbb{R}^d)$ $(d \geq 1)$ into itself which can also be defined on the dual space $\mathcal{S}'(\mathbb{R}^d)$ as the adjoint operator $\tilde{\mathcal{F}}^*$.

On the opposite, in the framework of white noise analysis, the definition of Fourier transform is instead a little less intuitive and its main peculiarity lies in the fact that it is constructed only as an operator from the space of Hida distributions - $(\mathcal{E}')$ - into itself. Nonetheless, since it represents a key component for the analysis of $\widetilde{G_{BMS}}$ covariant free fields and of their equations of motion we shall now introduce it following chapter 11 of [6]

Definition 4.7. We call **S-transform** of $\Psi \in (\mathcal{E}')$ the functional $\Psi^S : C^{\infty}(\mathbb{S}^2) \to \mathbb{C}$ such that

$$\Psi^S\left(\alpha(\zeta, \overline{\zeta})\right) = \langle\langle \Psi, : e^{\left(\cdot, \alpha(\zeta,\overline{\zeta})\right)} : \rangle\rangle,$$

where $\alpha(\zeta,\overline{\zeta}) \in C^\infty(\mathbb{S}^2)_{\mathbb{C}}$ and $\langle\langle,\rangle\rangle$ stands for the pairing between $(\mathcal{E})$ and $(\mathcal{E}')$.

We call **Fourier transform** the continuous linear operator $\mathcal{F} : (\mathcal{E}') \to (\mathcal{E}')$ such that $\widehat{\Psi} \doteq \mathcal{F}\Psi$ is the unique element in $(\mathcal{E}')$ satisfying

$$\widehat{\Psi}^s\left(\alpha(\zeta,\overline{\zeta})\right) = \langle\langle\Psi, e^{-i\left(\cdot,\alpha(\zeta,\overline{\zeta})\right)}\rangle\rangle = \Psi^s(-i\alpha(\zeta,\overline{\zeta}))e^{-\frac{1}{2}\left|\left|\alpha(\zeta,\overline{\zeta})\right|\right|_{L^2}}, \tag{42}$$

where $||,||_{L^2}$ stands for the norm in $L^2(\mathbb{S}^2)$.

Equivalently the Fourier transform is the unique continuous linear operator from $(\mathcal{E}')$ into itself such that for any $\alpha(\zeta,\overline{\zeta}) \in C^\infty(\mathbb{S}^2)$.

$$\mathcal{F}\widetilde{\mathcal{D}}_{\alpha(\zeta,\overline{\zeta})} = i\widetilde{Q}_{\alpha(\zeta,\overline{\zeta})}\mathcal{F}, \quad \mathcal{F}\widetilde{Q}_{\alpha(\zeta,\overline{\zeta})} = i\widetilde{\mathcal{D}}_{\alpha(\zeta,\overline{\zeta})}\mathcal{F}, \tag{43}$$

where $\widetilde{\mathcal{D}}$ is defined as in definition 4.6 and $\widetilde{Q}$ as in definition 4.4.

The above definition apparently put us into the position to rewrite in terms of derivative operators the equations of motion for a $\widetilde{G_{BMS}}$ massive or massless scalar field which, according to lemma 4.2, 4.3 and 4.4, are given by (39) and (40). Interpreting the field $\psi \in (\mathcal{E})$ as an element in $(\mathcal{E}')$ by means of the continuous inclusion of $(\mathcal{E})$ in $(\mathcal{E}')$, these latter equations become:

$$\widetilde{\mathcal{D}}_{\alpha(\zeta,\overline{\zeta})}\widehat{\psi} = 0, \quad \forall\alpha(\zeta,\overline{\zeta}) \in ST \tag{44}$$

$$\eta^{\mu\nu}\widetilde{\mathcal{D}}_{e_\mu}\widetilde{\mathcal{D}}_{e_\nu}\widehat{\psi} = \begin{cases} 0 & \text{for the little group } \Delta \\ m^2\widehat{\psi} & \text{for the little group } SU(2) \end{cases}, \tag{45}$$

where $\widehat{\psi} \in (\mathcal{E}')$.

It is imperative to underline a key aspects of the above differential equations: although (45) is similar to the well-known Klein-Gordon equation in Minkowski spacetime, $\eta^{\mu\nu}\widetilde{\mathcal{D}}_{e_\mu}\widetilde{\mathcal{D}}_{e_\nu}$, is not a symmetric operator on $(\mathcal{E}')$ contrary to $\eta^{\mu\nu}\partial_\mu\partial_\nu$ on $\mathcal{S}(\mathbb{R}^d)$. This difference is rather important since, as we shall see in the next section, if we wish to interpret the equations of motion of our fields as the extremum of a suitable (Lagrangian) functional, than the defining operator must be symmetric.

To avoid such a problem, we shall now exploit a new kind of transformation which has been discussed in [6, 35]. Still referring to our specific scenario the following holds:

Definition 4.8. We call **Fourier-Gauss** transform the continuous linear operator $\mathcal{G}_{a,b} : (\mathcal{E}) \to (\mathcal{E})$ such that, being $a, b \in \mathbb{C} - \{0\}$,

$$\mathcal{G}_{a,b}\psi(\beta) = \int_{\mathcal{E}'} \psi(a\beta' + b\beta)d\mu(\beta'). \quad \forall\psi \in (\mathcal{E}) \tag{46}$$

Furthermore, for all $a, b \in \mathbb{C} - \{0\}$ and for all $\alpha(\zeta,\overline{\zeta}) \in C^\infty(\mathbb{S}^2)$, it holds that

$$\mathcal{G}_{a,b}\mathcal{D}_{\alpha(\zeta,\overline{\zeta})} = b^{-1}\mathcal{D}_{\alpha(\zeta,\overline{\zeta})}\mathcal{G}_{a,b}, \tag{47}$$

$$\mathcal{G}_{a,b}Q_{\alpha(\zeta,\overline{\zeta})} = a^2b^{-1}\mathcal{D}_{\alpha(\zeta,\overline{\zeta})}\mathcal{G}_{a,b} + bQ_{\alpha(\zeta,\overline{\zeta})}\mathcal{G}_{a,b}, \tag{48}$$

where $\mathcal{D}_\eta$ and Q_η are the derivative and multiplication operators respectively introduced in definition 4.6 and 4.4. Moreover, bearing in mind the spaces $\left((\mathcal{E})_p, ||,||_p\right)$ as introduced in (31), if $a^2+b^2=1$ and $|b|=1$, then $||\mathcal{G}_{a,b}\psi(\beta)||_p = ||\psi(\beta)||_p$ for all $\psi \in (\mathcal{E})_p$ and for all $p \geq 0$.

We seek now to single out a preferred $\mathcal{G}_{a,b}$ within the set of Fourier-Gauss transforms parametrized by the complex numbers a, b. The criterion, we shall refer to, consists of requiring that the kernel of the operator $\eta^{\mu\nu}\mathcal{D}_{e_\mu}\mathcal{D}_{e_\nu}$ is mapped into the kernel of a new but symmetric operator. Bearing in mind that, for a linear operator, such a condition coincides with the request of self-adjointness, the following proposition holds:

Proposition 4.3. There are only two Fourier-Gauss transforms, namely $\mathcal{G}_{\sqrt{2},i}$ and $\mathcal{G}_{\sqrt{2},-i}$, such that $\mathcal{G}_{a,b}\eta^{\mu\nu}Q_{e_\mu}Q_{e_\nu} = A(a,b)\mathcal{G}_{a,b}$ where $A(a,b)$ is a linear continuous selfadjoint operator on $(\mathcal{E})$ which admits an extension to a unitary operator on $\left(L^2\right)$. Furthermore

$$A(\sqrt{2},\pm i) = \pm i\eta^{\mu\nu}\left(Q_{e_\mu} - 2\mathcal{D}_{e_\mu}\right)\left(Q_{e_\nu} - 2\mathcal{D}_{e_\nu}\right),$$

where the plus stands for $b=-i$ whereas the minus for $b=i$.

Proof. According to definition 4.8, for any non vanishing $a, b \in \mathbb{C}$ the Fourier-Gauss transform is a continuous linear operator from $(\mathcal{E})$ into itself such that, exploiting (47),

$$\mathcal{G}_{a,b}\eta^{\mu\nu}Q_{e_\mu}Q_{e_\nu} = \eta^{\mu\nu}\left[a^4b^{-2}\mathcal{D}_{e_\mu}\mathcal{D}_{e_\nu} + \right.$$
$$\left. + b^{-2}Q_{e_\mu}Q_{e_\nu} + a^2\left(\mathcal{D}_{e_\mu}Q_{e_\nu} + Q_{e_\mu}\mathcal{D}_{e_\nu}\right)\right]\mathcal{G}_{a,b}.$$

In order to realize when

$$A(a,b) = \eta^{\mu\nu}\left[a^4b^{-2}\mathcal{D}_{e_\mu}\mathcal{D}_{e_\nu} + b^{-2}Q_{e_\mu}Q_{e_\nu} + a^2\left(\mathcal{D}_{e_\mu}Q_{e_\nu} + Q_{e_\mu}\mathcal{D}_{e_\nu}\right)\right]$$

is self-adjoint on $(\mathcal{E})$ we refer to lemma 4.5 and to the relation $Q^*_{\alpha(\zeta,\overline{\zeta})} = Q_{\alpha(\zeta,\overline{\zeta})}$ on $(\mathcal{E})$ for any $\alpha(\zeta,\overline{\zeta}) \in C^\infty(\mathbb{S}^2)$ according to which:

$$A^*(a,b) = \eta^{\mu\nu}\left[a^4b^{-2}\mathcal{D}_{e_\mu}\mathcal{D}_{e_\nu} + \left(a^4b^{-2} + b^2 + 2a^2\right)Q_{e_\mu}Q_{e_\nu} + \right.$$
$$\left. - \left(a^4b^{-2} + a^2\right)\left(Q_{e_\mu}\mathcal{D}_{e_\nu} + \mathcal{D}_{e_\mu}Q_{e_\nu}\right)\right].$$

Thus, on $(\mathcal{E})$, $A^*(a,b) = A(a,b)$ iff $a^2 = -2b^2$. If we require that $\mathcal{G}_{a,b}$ could also be extended to a unitary operator on $\left(L^2\right)$, then, according to definition 4.8, we also impose $|b| = 1$ and $a^2 + b^2 = 1$; it implies that $b = \pm i$ and $a = \pm\sqrt{2}$.

To conclude we refer to theorem 11.28 in [6] and remarks below according to which $\mathcal{G}_{a,b} = \mathcal{G}_{c,d}$ iff $a = \pm c$ and $b = d$. Thus $\mathcal{G}_{\sqrt{2},\pm i} = \mathcal{G}_{-\sqrt{2},\pm i}$ on $(\mathcal{E})$. □

Remark 4.3. The arbitrariness in the choice of the Fourier-Gauss transform which arises from the previous theorem is only apparent. If we exploit

theorem 11.30 in [6], according to which, for any $a, b, c, d \in \mathbb{C} - \{0\}$, $\mathcal{G}_{c,d}\mathcal{G}_{a,b} = \mathcal{G}_{\pm\sqrt{a^2+b^2c^2},bd}$, we end up with

$$\mathcal{G}_{\sqrt{2},i} = \mathcal{G}^{-1}_{\sqrt{2},-i},$$

and viceversa.
Thus, whatever choice we shall perform, the other Fourier-Gauss transform is the inverse. Furthermore, since, according to the previous proposition, $A(\sqrt{2}, i) = -A(\sqrt{2}, -i)$ it is immediate to conclude that $\psi(\beta) \in Ker\left(A(\sqrt{2}, i)\right) \subset (\mathcal{E})$ iff $\psi(\beta) \in Ker\left(A(\sqrt{2}, -i)\right)$. For this reason we are entitled to deal only with one of the two choices for the Fourier-Gauss transform and, from now, $\mathcal{G}$ will stand for $\mathcal{G}_{\sqrt{2},i}$ whereas $\mathcal{G}^{-1} = \mathcal{G}_{\sqrt{2},-i}$.

To conclude the section we summarize the latter results i.e., if we start from (40) and (39) and if we perform the Fourier-Gauss transform $\mathcal{G}$, we end up with the following equations of motion for a $\widetilde{G_{BMS}}$ massive or massless real scalar field:

$$\left(-2\mathcal{D}_{\alpha(\zeta,\overline{\zeta})} + Q_{\alpha(\zeta,\overline{\zeta})}\right)\psi^G(\beta) = 0 \quad \forall\alpha(\zeta,\overline{\zeta}) \in ST \tag{49}$$

$$\eta^{\mu\nu}\left(-2\mathcal{D}_{e_\mu} + Q_{e_\mu}\right)\left(-2\mathcal{D}_{e_\nu} + Q_{e_\nu}\right)\psi^G(\beta) = \left\{\begin{array}{ll} 0 & \text{for } \Delta \\ m^2\ \psi^G(\beta) & \text{for } SU(2) \end{array}\right., \tag{50}$$

where $\psi^G(\beta) = \int\limits_{\mathcal{E}'} d\mu(\beta')\psi(\sqrt{2}\beta' + i\beta)$.

Remark 4.4. An interesting though, to a certain extent, heuristic comment concerning (50) arises if we write the Klein-Gordon equation of motion for a massless scalar field ψ in Minkowski spacetime M^4 starting from

$$L = \frac{1}{2}\int\limits_{M^4} d\mu(x^\rho)(-2\partial^\mu - x^\mu)\widetilde{\psi}(x^\rho)(-2\partial_\mu - x_\mu)\widetilde{\psi}(x^\rho).$$

Here $d\mu(x^\mu) = e^{-\frac{\delta^{\mu\nu}x_\mu x_\nu}{2}}d^4x$ is the standard Gaussian measure on $\mathbb{R}^4$ and $\widetilde{\psi}(x^\rho) = e^{\frac{\delta^{\mu\nu}x_\mu x_\nu}{4}}\psi(x^\rho)$ is the (rescaled) real scalar field; thus L is simply a rewriting of the usual Klein-Gordon Lagrangian and the associated equations of motion becomes:

$$\eta^{\mu\nu}(-2\partial_\mu + x_\mu)(-2\partial_\nu + x_\nu)\widetilde{\psi}(x^\rho) = 0.$$

A direct inspection of this equation shows a clear resemblance with (50) which confirms the rigorously proved correspondence between the Poincaré and the $\widetilde{G_{BMS}}$ massless real scalar fields (still see [13]).

5 The Lagrangian and Hamiltonian formulation of $\widetilde{G_{BMS}}$ scalar field theory.

In the previous discussions we have developed the covariant approach to $\widetilde{G_{BMS}}$ field theory exploiting the lone requirement that a free field is a suitably chosen function(al) which transforms under a unitary and irreducible representation of the full symmetry group. This perspective has allowed us not only to correctly identify the kinematical datum of a $\widetilde{G_{BMS}}$ field theory but, by means of functional analysis techniques, also the dynamic of these fields. Nonetheless the derivation of the $\widetilde{G_{BMS}}$ equations of motion (even limiting ourselves to the real scalar case) is still unsatisfactory for two main reasons; the first consists of the absence of any interaction which are a key cornerstone if one wish to develop a complete $\widetilde{G_{BMS}}$ field theory. Furthermore, from an holographic perspective, one would like to demonstrate the existence of an holographic mapping not only for free fields but also for the interacting ones and, in particular, we refer to gauge theories. To this avail it is imperative to derive the $\widetilde{G_{BMS}}$ equations of motion from a variational principle and in particular we wish to consider such a problem both in a Lagrangian and in an Hamiltonian framework for the "working example"of the covariant real (massless or massive) scalar field. The steps we will perform are the following: first we construct a suitable "Lagrangian" functional whose extremum provides (49) and (50) and then we derive the Hamiltonian function by means of standard techniques.

Remark 5.1. In order to construct the above mentioned Lagrangian, the starting point consists of introducing a suitable space of kinematically allowed configurations. In an infinite dimensional setting, there are two commonly accepted and widely exploited choices: the tangent bundle and the first jet bundle. In the latter case we should deal with equivalence classes of sections of an associated bundle over $\mathcal{E}'$. Such a road could be pursued within our framework following the definition 4.2 for a covariant $\widetilde{G_{BMS}}$ field though the characterization of a jet over the space of distribution over $\mathbb{S}^2$ is rather tricky.

On the other hand it is more convenient to our aims to follow the former case i.e. we will identify a tangent bundle over a suitable space of functions and an associated Lagrangian.

In the setting proper of $\widetilde{G_{BMS}}$ covariant field theory we deal with, the natural configuration space we have exploited up to now is a Fréchet manifold i.e. $(\mathcal{E})$ the space of Hida testing functionals. It is still possible to associate to it a notion of tangent space: we first need to recognize that $(\mathcal{E})$, constructed as in proposition 4.1, is an abelian ILH group and thus we are entitled to follow [29] and to define $T(\mathcal{E}) = \bigcap_p T(\mathcal{E})_p$. Furthermore, being $(\mathcal{E})$ abelian, we may also conclude that $T(\mathcal{E}) = (\mathcal{E}) \times (\mathcal{E})$.

A further option which arises and which follows more closely the usual setting of Poincaré invariant field theories consists of reminding that, according

to proposition 4.1, the set of Hida testing functionals is continuously included in (L^2). Thus we can enlarge the space of kinematical configurations to (L^2). Such a choice is not only a mere convenience if we bear in mind that both the canonical and the covariant $\widetilde{G_{BMS}}$ field have been originally introduced as function(als) on Hilbert space of square integrable functions.
As a matter of fact all operators involved in the construction of the previous sections, namely $Q_{\alpha(\zeta,\overline{\zeta})}$ and $\mathcal{D}_{\alpha(\zeta,\overline{\zeta})}$ admits a unique continuous extension from their natural space of definition - $(\mathcal{E})$ - to (L^2) for any $\alpha(\zeta,\overline{\zeta}) \in C^\infty(\mathbb{S}^2)$ and, as outlined in proposition 4.3, also the Fourier-Gauss transform can be continuously extended to a unitary operator on (L^2).

Bearing in mind these remarks we shall work in this section with (L^2) whose Hilbert structure allows us an easier identification of the Lagrangian function; we will point out in the end that the result holds as well in $(\mathcal{E})$.

Starting from these premises, we shall now solve the inverse "Lagrangian" problem i.e. we shall start seeking for a functional $L : (L^2) \to \mathbb{R}$ whose extremum is (49) and (50). The strategy we follow consists on ignoring at the beginning (49) requiring only that our $\widetilde{G_{BMS}}$ real massive or massless scalar field satisfies (50). To this avail, we shall employ a standard technique due to Vainberg [36, 37]. Let us remind the reader that, given a Banach space X and its dual space X', an operator $F : X \to X'$ is called *potential* on some subset $H \subset E$ iff it exists a functional f on X such that $F(x) = \nabla f(x)$ where ∇ is the gradient[14] of the functional f.

Bearing in mind such a definition, the following theorem holds (we refer to §5 in [36] for the proof):

Theorem 5.1. [Vainberg] Suppose that X is a Banach space with norm $||,||$ and that $F : X \to X'$ admits a Gateaux differential $D_hF(x)$ for all x lying in a norm induced ball $B_r(x_0)$ centered in a point $x_0 \in X$ and of arbitrary but fixed radius r. Suppose also that the functional $(D_hF(x), h')$ is continuous in the variable $x \in B_r(x_0)$. Then F is potential in $B_r(x_0)$ iff $(D_hF(x), h') = (D_{h'}F(x), h)$ for all $h, h' \in X$ where $(,)$ represents the natural pairing between X and X'.

It is straightforward now to realize from the statement of this latter theorem why we considered unsatisfactory the Fourier transform in order to formulate the $\widetilde{G_{BMS}}$ equations of motion in a evolutionary form. If we wish to follow the "traditional" road of quantum field theory over Minkowski spacetime and, if we look for a formulation of the $\widetilde{G_{BMS}}$ equations of motion (44) and (45) as a variational problem, we realize that, although the operator $\eta^{\mu\nu}\mathcal{D}_{e_\mu}\mathcal{D}_{e_\nu}$ admits a Gateaux differential and $\eta^{\mu\nu}\mathcal{D}_{e_\mu}\mathcal{D}_{e_\nu}\psi(\beta)$ is continuous in the variable β, it fails to be symmetric. For a linear operator F, such as the one we are dealing with,

[14]We remember that an operator $F : X \to X'$ is called the gradient of a functional f if f admits along all directions on X a Gateaux derivative which, furthermore, must coincide with F.

even though we should restrict ourselves from the space $(\mathcal{E}')$, where (44) and (45) are naturally defined, to $\left(L^2\right)$, the symmetry condition would still imply:

$$(D_h F(x), h') = (F(h), h') = (h, F(h')) = (D_{h'} F(x), h)\,, \tag{51}$$

i.e. F is self adjoint. It does not hold in our scenario since, exploiting lemma 4.5, one can see that

$$\eta^{\mu\nu}\mathcal{D}_{e_\mu}\mathcal{D}_{e_\nu}(\beta) - \eta^{\mu\nu}\mathcal{D}^*_{e_\mu}\mathcal{D}^*_{e_\nu}(\beta) = \eta^{\mu\nu}\left[\mathcal{D}_{e_\mu}Q_{e_\nu} + Q_{e_\mu}\mathcal{D}_{e_\nu} - Q_{e_\mu}Q_{e_\nu}\right].$$

This is the first real big difference from the canonical procedure for a scalar theory formulated in Minkowski background. Up to now, besides the complicated techniques of white noise distribution theory, we have basically repeated at least conceptually the same steps we would have performed in a Poincaré invariant setup. At this stage, instead, we face the serious obstruction of Vainberg theorem and this one is the main reasons why we shall adopt the Fourier-Gauss transform $\mathcal{G}$ as in proposition 4.3. Within this framework the following holds

Lemma 5.1. Referring to Q and to D as the (unique continuous) extension of the multiplication and derivative operator from $(\mathcal{E})$ to $\left(L^2\right)$ and referring to e_μ, e_ν as $\left\{Y_{00}(\zeta,\overline{\zeta}), ..., Y_{11}(\zeta,\overline{\zeta})\right\}$, then the operator

$$\eta^{\mu\nu}\left(Q_{e_\mu} - 2\mathcal{D}_{e_\mu}\right)\left(Q_{e_\nu} - 2\mathcal{D}_{e_\nu}\right) : \left(L^2\right) \to \left(L^2\right)$$

is potential and the unique functional $L_{dyn} : \left(L^2\right) \to \mathbb{R}$, whose value in $\psi_0 \in \left(L^2\right)$ is L_0, is

$$L_{dyn}(\psi) = L_0 + \int_0^1 dt\, \langle\langle \eta^{\mu\nu}\left(Q_{e_\mu} - 2\mathcal{D}_{e_\mu}\right)\left(Q_{e_\nu} - 2\mathcal{D}_{e_\nu}\right) \\ \left(\psi_0 + t(\psi - \psi_0)\right), \psi - \psi_0\rangle\rangle_{(L^2)}, \tag{52}$$

where $\langle\langle,\rangle\rangle_{(L^2)}$ is the internal product [15] on $\left(L^2\right)$.

Proof. Identifying the Hilbert space $\left(L^2\right)$ with its dual by means of the Riesz theorem, the operator $A = \eta^{\mu\nu}\left(Q_{e_\mu} - 2\mathcal{D}_{e_\mu}\right)\left(Q_{e_\nu} - 2\mathcal{D}_{e_\nu}\right)$ is a map from $\left(L^2\right)$ to $\left(L^2\right)'$. It admits a continuous Gateaux derivative[16] for all $\psi \in \left(L^2\right)$ and along all directions $\psi' \in \left(L^2\right)$ since, per definition (41) and, being A linear,

$$\mathcal{D}_{\psi'}\left[\eta^{\mu\nu}\left(Q_{e_\mu} - 2\mathcal{D}_{e_\mu}\right)\left(Q_{e_\nu} - 2\mathcal{D}_{e_\nu}\right)\right]\psi = \eta^{\mu\nu}\left(Q_{e_\mu} - 2\mathcal{D}_{e_\mu}\right)\left(Q_{e_\nu} - 2\mathcal{D}_{e_\nu}\right)\psi'.$$

[15] We adopt the symbol $\langle\langle,\rangle\rangle$ which stands for the natural pairing between $(\mathcal{E})$ and $(\mathcal{E}')$ because it is subject to the compatibility condition (32) according to which it coincides with the internal product on $\left(L^2\right)$ when we evaluate $\langle\langle\phi,\phi'\rangle\rangle$ with $\phi \in (\mathcal{E})$ and $\phi' \in \left(L^2\right)$.

[16] The definition Gateaux derivative on a functional from $\left(L^2\right)$ to $\mathbb{R}$ is a straightforward adaptation of definition 4.6. For this reason, we feel that, for the economy of the paper, it is useless to introduce an additional symbol and we will use also in this case $\mathcal{D}$. The associated pedex will univocally distinguish between the different cases.

Furthermore, on $\left(L^2\right)$ the operator under analysis is according to proposition 4.3 selfadjoint thus symmetric. The hypotheses of Vainberg theorem are met and $\eta^{\mu\nu}\left(Q_{e_\mu}-2\mathcal{D}_{e_\mu}\right)\left(Q_{e_\nu}-2\mathcal{D}_{e_\nu}\right)$ is potential. Uniqueness of the functional - i.e. (52) - whose gradient satisfies the equation (49) is now a direct consequence of Vainberg theorem which grants us that

$$\mathcal{D}_{\psi'}L(\psi)=\langle\langle\eta^{\mu\nu}\left(Q_{e_\mu}-2\mathcal{D}_{e_\mu}\right)\left(Q_{e_\nu}-2\mathcal{D}_{e_\nu}\right)\psi,\psi'\rangle\rangle_{(L^2)}.$$

Thus for any ψ in a ball $D=\left\{\psi\in\left(L^2\right)\mid\langle\langle\psi-\psi_0\rangle\rangle_{(L^2)}<r\right\}$ centered in ψ_0 and of fixed radius r, and for any $t\in[0,1]$ the last equality translates as

$$\frac{d}{dt}L\left(\psi_0+t(\psi-\psi_0)\right)=$$
$$=\langle\langle\eta^{\mu\nu}\left(Q_{e_\mu}-2\mathcal{D}_{e_\nu}\right)\left(Q_{e_\nu}-2\mathcal{D}_{e_\nu}\right)\left(\psi_0+t(\psi-\psi_0)\right),\psi-\psi_0\rangle\rangle.$$

An integration in the t variable shows that (52) is the unique functional whose gradient is our equation $\eta^{\mu\nu}\left(Q_{e_\mu}-2\mathcal{D}_{e_\mu}\right)\left(Q_{e_\nu}-2\mathcal{D}_{e_\nu}\right)\psi(\beta)=0$. $\Box$

Remark 5.2. Setting the initial condition as $\psi_0=0$, $L_0=0$ and adding the mass term whenever we wish to deal with a massive $\widetilde{G_{BMS}}$ real scalar field, then (52) becomes:

$$\begin{aligned}L_{KG}(\psi)=\frac{1}{2}\langle\langle\left[\eta^{\mu\nu}\left(Q_{e_\mu}-2\mathcal{D}_{e_\mu}\right)\left(Q_{e_\nu}-2\mathcal{D}_{e_\nu}\right)+m^2\right]\psi,\psi\rangle\rangle_{(L^2)}=\\ \frac{1}{2}\int\limits_{\mathcal{E}'}d\mu(\beta)\eta^{\mu\nu}\left[-4\mathcal{D}_{e_\mu}\psi(\beta)\mathcal{D}_{e_\nu}\psi(\beta)+\right.\\ \left.(\beta,e_\mu)(\beta,e_\nu)\psi^2(\beta)+4(\beta,e_\mu)\psi(\beta)\mathcal{D}_{e_\nu}\psi(\beta)+m^2\psi^2(\beta)\right],\end{aligned}\tag{53}$$

where in the last equality we have exploited the definition of multiplication operator and lemma 4.5 whereas (β,e_μ) still stands for the canonical pairing between $\mathcal{E}'$ and $C^\infty(\mathbb{S}^2)$.

We will refer to this term as the *Klein-Gordon part of the Lagrangian* for the $\widetilde{G_{BMS}}$ massive or massless scalar field. It is also imperative to underline that the above methods can be fully applied also to non scalar $\widetilde{G_{BMS}}$ field without any substantial modifications in the reasoning and in the demonstrations. This still confirms that we are working with a specific field only for the sake of simplicity and of clarity nonetheless without losing in generality.

We face now the last obstacle i.e. we need also to implement (50). A direct inspection shows that (50) is a family of constraints on the covariant fields and thus we seek to implement it in terms of Lagrange multipliers:

Proposition 5.1. The functional $L:\left(L^2\right)\to\mathbb{R}$ whose associated Euler equations are (49) and (50) is

$$L(\psi,\lambda_i)=L_{KG}(\psi)+\sum_i\int\limits_{\mathcal{E}'}d\mu(\beta)\frac{\lambda_i}{2}(\beta)\left[\left(-2\mathcal{D}_{e_i}+Q_{e_i}\right)\psi(\beta)\right]^2,\tag{54}$$

where L_{dyn} is (5.2), $e_i = \left\{Y_{lm}(\zeta,\overline{\zeta})\right\}_{l>1}$ whereas $\lambda_i(\beta) \in \left(L^2\right)$ are suitable Lagrange multipliers. Furthermore, as previously, the operator Q refers to the unique continuous extension to $\left(L^2\right)$ of the corresponding Gateaux derivative and multiplication operator on $(\mathcal{E})$.

Proof. The first step in the demonstration consists of showing that an element $\psi(\beta) \in \left(L^2\right)$ satisfies $\left(-2\mathcal{D}_{\alpha(\zeta,\overline{\zeta})} + Q_{\alpha(\zeta,\overline{\zeta})}\right)\psi(\beta) = 0$ for all $\alpha(\zeta,\overline{\zeta}) \in ST$ iff $\left(-2\mathcal{D}_{e_i} + Q_{e_i}\right)\psi(\beta) = 0$ for each e_i. This statement straightforwardly holds since, according to theorem 2.2, ST is the closed set of real linear combinations of the real spherical harmonics with $l > 1$ and since the operator $-2\mathcal{D} + Q$ seen as a map from $C^\infty(\mathbb{S}^2) \times \left(L^2\right) \to \left(L^2\right)$ mapping the pair $(\alpha(\zeta,\overline{\zeta}), \psi(\beta))$ into $\left(-2\mathcal{D}_{\alpha(\zeta,\overline{\zeta})} + Q_{\alpha(\zeta,\overline{\zeta})}\right)\psi(\beta)$ is linear in the first argument.

The remaining part of the proof will be structured as follows: we will calculate the variation with respect to ψ of a generic functional

$$L(\psi) = \int\limits_{\mathcal{E}'} d\mu(\beta)\mathcal{L}(\beta, \psi(\beta), \mathcal{D}_{\beta'}\psi(\beta)),$$

where $\mathcal{L} : \left(L^2\right) \to \left(L^2\right)$ is a "density" depending both on the fields and on its derivative along any direction. The final result will be the "Euler-Lagrange" equation associated to a functional defined on a space endowed with a Gaussian measure. Eventually we will apply the result to (54).

Let us thus perform the following variation: pick any $\phi(\beta) \in \left(L^2\right)$, then

$$\langle\langle \frac{\delta L}{\delta\psi}, \phi(\beta)\rangle\rangle = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[L(\beta, \psi + \epsilon\phi, \mathcal{D}_{\beta'}(\psi + \epsilon\phi)) - L(\beta, \psi, \mathcal{D}_{\beta'}\psi)\right] =$$

$$= \int\limits_{\mathcal{E}'} d\mu(\beta)\ \mathcal{D}_\psi\mathcal{L}(\beta, \psi, \mathcal{D}_{\beta'}\psi)\phi(\beta) + \int\limits_{\mathcal{E}'} d\mu(\beta)\ \mathcal{D}_{\mathcal{D}_{\beta'}\psi}\mathcal{L}(\beta, \psi, \mathcal{D}_\beta\psi)\mathcal{D}_{\beta'}\phi(\beta).$$

The second element in the right hand side of the last equality can be written in the more convenient form

$$\langle\langle \mathcal{D}_{\mathcal{D}_{\beta'}\psi}\mathcal{L}(\beta, \psi, \mathcal{D}_\beta\psi), \mathcal{D}_{\beta'}\phi(\beta)\rangle\rangle = \langle\langle \mathcal{D}^*_{\beta'}\left[\mathcal{D}_{\mathcal{D}_{\beta'}\psi}\mathcal{L}(\beta, \psi, \mathcal{D}_\beta\psi)\right], \phi(\beta)\rangle\rangle =$$

$$= \langle\langle \left(-\mathcal{D}_{\beta'} + Q_{\beta'}\right)\mathcal{D}_{\mathcal{D}_{\beta'}\psi}\mathcal{L}(\beta, \psi, \mathcal{D}_\beta\psi), \phi(\beta)\rangle\rangle,$$

where $\mathcal{D}^*_{\beta'}$ is the adjoint derivative operator and where we have exploited the relation $\mathcal{D}^*_{\beta'} + \mathcal{D}_{\beta'} = Q_{\beta'}$ as in lemma 4.5.

Thus in order for the variation of (54) to vanish for any choice of $\phi(\beta)$, we end up with the following Euler-Lagrange equation:

$$\mathcal{D}_\psi\mathcal{L}(\beta, \psi, \mathcal{D}_{\beta'}\psi) - \left(\mathcal{D}_{\beta'} - Q_{\beta'}\right)\mathcal{D}_{\mathcal{D}_{\beta'}\psi}\mathcal{L}(\beta, \psi, \mathcal{D}_\beta\psi) = 0, \tag{55}$$

where the term with the multiplication operator is a feature typical due to the presence of a Gaussian measure μ on $\mathcal{E}'$.

This formula can be straightforwardly extended when, as in the scenario under consideration, the functional depends upon more than one field. Thus a

straightforward application of (55) in (54) shows that the variation of functional for the scalar $\widetilde{G_{BMS}}$ field with respect to the Lagrange multipliers provides that $\left(-2\mathcal{D}_{e_i} + Q_{e_i}\right)\psi(\beta) = 0$ for all e_i whereas a variation with respect to ψ provides, once the constraints are imposed, equation (40). $\square$

Remark 5.3. In the wake of the above proposition, it is natural and at same time imperative to wonder ourselves whether we are titled to really refer to (54) as the Lagrangian of our system. As a matter of fact a direct inspection of (54) shows that the functional under analysis can also be interpreted as map $L : T\left(L^2\right) \to \mathbb{R}$ which associates to the pair $(\psi, \mathcal{D}_{e_0}\psi) \in T\left(L^2\right) \equiv \left(L^2\right) \times \left(L^2\right)$ the expression (54). Here $\mathcal{D}_{e_0}$ represents the natural counterpart for the time derivative of a covariant field in Minkowski spacetime.

Furthermore, bearing in mind that the Gateaux derivative along any direction on $\mathcal{E}'$ is a continuous map from $(\mathcal{E})$ into itself, it is immediate to realize that the above interpretation for (54) holds also if we refer to Hida testing functionals. Thus $L(\psi, \lambda_i)$ can also be taught as the Lagrangian for the $\widetilde{G_{BMS}}$ scalar field on $T\left(\mathcal{E}\right)$.

Having solved the inverse Lagrangian problem, we are now apparently in position to formulate the free $\widetilde{G_{BMS}}$ field theory in an Hamiltonian framework. While the Lagrangian analysis is best performed in the tangent space of a suitably chosen configuration space, the Hamiltonian counterpart is naturally developed with the tools proper of symplectic geometry. Since it will also play a fundamental role in the forthcoming analysis we will first choose a symplectic space and the natural obvious choice is the cotangent bundle over our configuration space. If one wishes to work directly with the the space of Hida testing functional, it is natural to resort again to the identification of $(\mathcal{E})$ with an abelian ILH which leads to identify $T\left(\mathcal{E}\right) = (\mathcal{E}) \times (\mathcal{E})$ and, per duality, $T^*\left(\mathcal{E}\right) = (\mathcal{E}) \times (\mathcal{E}')$. On the opposite we will choose as configuration space $\left(L^2\right)$ which, bearing in mind that $T\left(L^2\right) = \left(L^2\right) \times \left(L^2\right)$, allows us to identify by Riesz theorem $T^*\left(L^2\right) = \left(L^2\right) \times \left(L^2\right)$.

As for the Lagrangian counterpart we will focus first on the cotangent bundle with an Hilbert structure remarking in the end that all the results can be applied also in the former case without any significant modification.

Definition 5.1. We call Γ, the Cartesian product $\left(L^2\right) \times \left(L^2\right)$ the *phase space* of a $\widetilde{G_{BMS}}$ real (massive or massless) scalar field associated to the configuration space $\left(L^2\right)$. If the latter is chosen as $(\mathcal{E})$ then $\Gamma = (\mathcal{E}) \times (\mathcal{E}')$.

Following nomenclatures of [18, 38]:

Proposition 5.2. The vector space Γ endowed with the continuous bilinear map (with respect to the product topology) $\Omega : \Gamma \times \Gamma \to \mathbb{R}$

$$\Omega\left((\psi_1, \Psi_1), (\psi_2, \Psi_2)\right) = \langle\langle \Psi_2, \psi_1 \rangle\rangle - \langle\langle \Psi_1, \psi_2 \rangle\rangle \tag{56}$$

is a symplectic vector space. Here $\langle\langle , \rangle\rangle$ is either the canonical pairing between $(\mathcal{E})$ and $(\mathcal{E}')$ or the internal product on $\left(L^2\right)$ depending on the chosen phase space.

Proof. We need only to show that Ω is a weakly non degenerate skew symmetric bilinear form. Independently from the two possible cases in the hypotheses, skew-symmetry is trivially verified whereas, in order to show the weak non degenerateness of Ω, we need to show that, calling $\nu_1 = (\psi_1, \Psi_1)$ and $\nu_2 = (\psi_2, \Psi_2)$, then $\Omega(\nu_1, \nu_2) = 0$ for any $\nu_2 \in \Gamma$ implies $\nu_1 = 0$. Choose $\nu_2 = (0, \Psi_2)$; then $\Omega(\nu_2, \nu_2) = \langle\langle \Psi_2, \psi_1 \rangle\rangle = 0$ for any choice of $\Psi_2 \in (\mathcal{E}')$. This is possible iff $\psi_1 = 0$. Similarly choose now $\nu_2 = (\psi_2, 0)$; accordingly $\Omega(\nu_2, \nu_1) = \langle\langle \Psi_1, \psi_2 \rangle\rangle = 0$ for any choice of $\psi_2 \in (\mathcal{E})$. This is achievable only if $\Psi_1 = 0$. $\square$

Remark 5.4. It is interesting to pinpoint that, if we resort to work on nuclear spaces such as $(\mathcal{E})$, we are constrained to deal only with Fréchet structures which, thus, forbids us to select a strongly non degenerate symplectic space which is the natural structure in finite dimensional dynamical systems. On the opposite, if we choose to work on the Hilbert spaces ,such as $\left(L^2\right)$, it is straightforward to realize, still thanks to Riesz theorem, that $\Omega^b : \Gamma \to \Gamma^*$, mapping $\nu \in \Gamma$ into the linear operator $\Omega(\nu) : \Gamma \to \mathbb{R}$ is an isomorphism and thus the symplectic form is *strongly non degenerate.*

Thus from now we will consider only the symplectic phase space $\left(T^*\left(L^2\right), \Omega\right)$.

Bearing in mind the above comments one immediately realize that the construction of the Hamiltonian function is not a straightforward calculation since (54) is a singular Lagrangian. Thus we need to resort to the theory of constraints and in particular to the algorithm developed by Gotay, Nester and Hinds in [39, 40] which is a geometrization and a generalization of the canonical Dirac-Bergman theory. In particular here we will adapt to our Hilbert configuration manifold the analysis of a Lagrangian system with Lagrange multipliers performed in [41, 42] for finite dimensional configuration spaces.

Since the constraints (50) are globally defined on $T\left(L^2\right)$, the first natural step consists of promoting the multipliers in (54) to dynamical variable thus switching from $T\left(L^2\right)$ to $TP \equiv T\left[\left(L^2\right) \times \left(L^2\right)^N\right]$ where $\left(L^2\right)^N$ means that we consider as many copies of $\left(L^2\right)$ as the number of needed Lagrange multipliers. In the case under consideration this is equal to the number of spherical harmonics with $l > 1$. Local coordinates on P are given by $(\psi, \mathcal{D}_{e_0}\psi, \lambda_i, \mathcal{D}_{e_0}\lambda_i)$ where now e_0 stands for the $l = 0$ spherical harmonic; it plays the role of the time direction in a Poincaré invariant theory. Reading (50) as a map $L : TP \to \mathbb{R}$, we can introduce the fiber derivative $\mathcal{F}L : TP \to T^*P$ such that

$$\langle\langle \psi', \mathcal{F}L(\psi) \rangle\rangle_{(L^2)} = \frac{d}{dt} L(\psi + t\psi')\bigg|_{t=0} = \mathcal{D}_{\psi'} L(\psi) = \langle\langle \mathcal{D}L, \psi' \rangle\rangle_{(L^2)}.$$

In local coordinates such a transformation becomes

$$\mathcal{F}L(\psi, \mathcal{D}_{e_0}\psi, \lambda_i, \mathcal{D}_{e_0}\lambda_i) = (\psi, \mathcal{D}_{\mathcal{D}\psi}L, \lambda_i, 0) = (\psi, 4\mathcal{D}_{e_0}\psi - 2Q_{e_0}\psi, \lambda_i, 0)\,. \quad (57)$$

The above equality simply restates that the Lagrangian function is not hyper-regular and thus the fiber derivative is not a diffeomorphism. Consequently we are obstructed to introduce the Hamiltonian as $H = E \circ \mathcal{F}L^{-1}$ where E represents the energy function

$$E = \langle\langle \psi, \mathcal{F}L(\psi)\rangle\rangle_{(L^2)} - L(\psi).$$

On the opposite we may still construct it implicitly on the image of $\mathcal{F}L(TP)$ as $H \circ \mathcal{F}L = E$ which is a reasonable definition iff for any two points $p, p' \in TP$ such that $\mathcal{F}L(p) = \mathcal{F}L(p')$ then $E(p) = E(p')$.

As discussed mainly in [40], such last condition is satisfied if the Lagrangian under analysis is *almost regular* i.e. $\mathcal{F}L$ is a submersion onto T^*P and, for any $p \in TP$, the fibers $(\mathcal{F}L)^{-1}\{\mathcal{F}L(p)\}$ are connected submanifolds of TP.

Proposition 5.3. The functional (54) is an almost regular Lagrangian.

Proof. The demonstration is divided in two parts: first we show that $\mathcal{F}L(TP)$ is a submersion and than we prove that the fibers are connected submanifolds.

In order to deal with the first assertion we exploit proposition 2.2 in chapter II §2 of [43] according to which a class C^p $(p \geq 0)$ morphism f between two manifolds of class C^p - X, Y - modelled over Banach spaces is a submersion at $x \in X$ iff it exists a chart (U, φ) at x and a second chart (V, ϕ) at $f(x) \in Y$ such that $\mathcal{D}f_{V,U}(\varphi(x))$ is surjective and the kernel splits

In the hypotheses of this proposition both TP and T^*P are Hilbert spaces which can be identified exploiting Riesz theorem. Thus, choosing any chart centerd at a point $p \in TP$, a direct inspection of (57) shows either that the fiber derivative is a surjection on its image either that the kernel of $\mathcal{F}L$ is the set of real linear combinations of vectors $(0, 0, 0, \mathcal{D}\lambda_i)$. Thus $Ker\,(\mathcal{F}L(TP))$ is isomorphic to $\left(L^2\right)^N$ and $TP = Ker(\mathcal{F}L) + M_1$ where $M_1 = \left(L^2\right) \times \left(L^2\right) \times \left(L^2\right)^N$ with $M_1 \cap Ker(\mathcal{F}L) = \{0\}$. This latter decomposition induces a natural map from TP into the Cartesian product $Ker(\mathcal{F}L) \times M_1$ which is a (toplinear) isomorphism and thus the kernel splits.

Concerning the second part of the demonstration, consider any point $q \in T^*(P)$ such that $q = \mathcal{F}L(\bar{p})$ with $\bar{p} \in TP$. Pick any two points - say p_1, p_2 lying in $\mathcal{F}L^{-1}(q)$. Referring to $\tau : TP \to P$ as the tangent bundle projection map and to $\pi : T^*P \to P$ as the cotangent bundle counterpart, we may conclude from the compatibility condition $\pi \circ \mathcal{F}L = \tau$ that $\tau(p_1) = \tau(p_2)$ i.e. $(\psi_1, \lambda_{i1}) = (\psi_2, \lambda_{i2})$. To conclude the demonstration it is sufficient now to exploit (57) and the hypothesis $\mathcal{F}L(p_1) = \mathcal{F}L(p_2)$ according to which

$$(\psi_1, 4(\mathcal{D}_{e_0}\psi)_1 - 2Q_{e_0}\psi_1, \lambda_{i1}, 0) = (\psi_2, 4(\mathcal{D}_{e_0}\psi)_2 - 2Q_{e_0}\psi_2, \lambda_{i2}, 0)\,.$$

It implies that the two points p_1, p_2 differ at most for an element in $Ker(\mathcal{F}L)$; thus the fibers are connected submanifolds. □

As a consequence of this last theorem, we know that the energy function in constant along the fibers of $\mathcal{F}L$ and thus it induces on the manifold M_1 a well defined Hamiltonian function as

$$H(\psi, \lambda_i, \Pi) = \frac{1}{2}\int\limits_{\mathcal{E}'} d\mu(\beta)\left[\Pi^2(\beta) - \sum_{k=1}^{3}\left[Q_{e_k}\psi(\beta) - 2\mathcal{D}_{e_k}\psi(\beta)\right]^2 - m^2\psi^2(\beta)\right] + \\ -\frac{1}{2}\sum_i \int\limits_{\mathcal{E}'} d\mu(\beta)\lambda_i(\beta)\left[\left(-2\mathcal{D}_{e_i} + Q_{e_i}\right)\psi(\beta)\right]^2, \tag{58}$$

where $\Pi = -2\mathcal{D}_{e_0}\psi(\beta) + Q_{e_0}\psi(\beta)$ is the conjugate momentum whereas e_k are the three spherical harmonics direction in $C^\infty(\mathbb{S}^2)$ with $l = 1$.

6 Comments and conclusions

The overall results of this paper could simply be summarized with the set phrase "the circle has been closed". Starting from [10], it was realized that the infinite dimensional nature of the supertranslations and of the supermomenta forces us to deal with $\widetilde{G_{BMS}}$ fields being functionals instead of the canonical functions proper of a Poincaré invariant theory over Minkowski spacetime or more generally of a quantum field theory over a curved background.

Consequently it appeared that only the purely group theoretical Wigner programme could shed some light on the kinematically and dynamically allowed configurations for a BMS invariant field theory living at future (or past) null infinity; the paradigm of equations of motion as an extremum out of a variational principle was thus a priori discarded.

Such an obstruction was previously gotten around exploiting the rigorous means of algebraic quantum field theory out of which some "holographic theorems" were proved. In this paper we wished to overcome the above deficiency ad we managed to associate to a scalar $\widetilde{G_{BMS}}$ field theory a genuine Hamiltonian system. To achieve such a goal we followed the path to rigorously define and analyze the covariant formulation of a $\widetilde{G_{BMS}}$ invariant theory. Within this framework each field arises as an element in a suitably constructed space of Hida testing functionals or, more generally, in its univocally associated Gelfand triplet.

This novel point of view lead us to a twofold result: as a first step we casted the equations of motion for a $\widetilde{G_{BMS}}$ field as suitable operators acting on the above mentioned space of Hida testing functionals. Afterwards, by a continuous extension, to its Hilbert space completion, we have shown that each equation of motion for a real massive or massless scalar field could be interpreted as the Euler-Lagrange equation of a suitable functional.
Alas, such a Lagrangian turned out not be hyperregular and thus the fiber derivative from the tangent to the cotangent space over the set of kinematical configuration is not a diffeomorphism. Exploiting the geometric description of the constraint algorithm originally due to Nester and Goaty for presymplectic

Lagrangian manifolds, we have nonetheless manged to show that on a suitable connected submanifold of the symplectic cotangent bundle, we could identify an Hamiltonian function.

Compared to Ashtekar and Streubel result, a direct inspection shows that, since our analysis starts from an intrinsic definition of a $\widetilde{G_{BMS}}$ field theory, it enlightens the contribution of the pure supertranslational component of $C^\infty(\mathbb{S}^2)$ which appeared to be partially neglected in [17]. Consequently we confirm the conclusions sketched already in [11] according to which the result in [17] encompasses mainly the datum from what we referred to as the Klein-Gordon component of the $\widetilde{G_{BMS}}$ dynamical system.

From a future perspective, one could claim that, on a physical ground, the results achieved put us into the position to discuss without further ado if an holographic correspondence between bulk and boundary (Yang-Mills) gauge theories really exists in an asymptotically flat spacetime. As already mentioned in the introduction the next direct step after our analysis starts from the results of section 4 and 5 leading to the development of symplectic techniques out of which we may construct a $\widetilde{G_{BMS}}$ interacting field theory.

From a mere holographic point of view, although it was more an underlying motivation for the whole line of research rather than for this specific paper, we can nonetheless comment that we have now better clarified, from the functional analytic point of view, the existence of the bulk to boundary correspondence for massless real scalar fields proved in [13]. In particular remark 4.4 outlined that the relevant operators, describing the dynamic of the field theory both in a flat background and at null infinity, are ultimately the same. As a side remark, one could also hope that such a line of thinking could shed some light on the problem, mentioned in the introduction, to construct a full holographic correspondence for massive free field. Within the "functional perspective" there is no apparent obstruction to relate massive fields on Minkowski and on its conformal boundary and thus the obstruction lies in developing a concrete geometrical way to project the data from the bulk to null infinity itself.

From a pure mathematical point of view it appears that the realization of BMS field theory as a dynamical system can be coherently and fully described in terms of white noise analysis. The only minor obstruction to the date consists in the "tangent bundle" approach. In a finite dimensional counterpart, it is common to formulate classical field theory in terms of jet bundles which allow to treat on the same ground time and spatial derivatives. Such a problem clearly arises also in a $\widetilde{G_{BMS}}$ framework where one wishes to encompass in a unique setting all the Gateaux derivatives along $\mathcal{E}'$-directions. Unfortunately, as outlined in section 4, covariant $\widetilde{G_{BMS}}$ fields are maps from $\mathcal{E}'$ into a suitable target space and the former is not a priori a Fréchet manifold but simply a locally convex topological space. Thus it appears to be rather difficult, or at least unknown to us, how to coherently introduce, within the $\widetilde{G_{BMS}}$ framework, the notion of (first) jet bundle; the most promising road within this direction lies in a sheaf theoretical formulation of the Hamiltonian theory though it would

possibly forbid us to deal with global issues addressing only the local ones. We will analyze in detail such a problem in a future paper.

Acknowledgements

The author is in great debt with V. Moretti, M. Carfora and O. Maj for several usefull discussions during the realization of this manuscript. The work has been supported partially by a grant from the Department of Theoretical and Nuclear Physics of Pavia University and partially by a grant from GNFM-INdAM (Istituto Nazionale Di Alta Matematica) under the project *"Olografia e spazitempo asintoticamente piatti: un approccio rigoroso"*.

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